The Cross-Classified Age-Period-Cohort Model as a Constrained Estimator*

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Abstract

Age-Period-Cohort (APC) models are designed to separate the independent effects of age, time periods, and cohort membership. However, such APC models suffer from an identification problem: there are no valid solutions because of the exact linear dependency among age, period, and cohort. Among methods proposed to address this problem, the cross-classified approach, including Cross-Classified Fixed Effects Models (CCFEM) and Cross-Classified Random Effects Models (CCREM), appears to solve the identification problem and to yield good estimates of the independent effects of age, period, and cohort groups. This paper assesses the validity and application scope of CCFEM and CCREM theoretically and illustrates their properties with simulations. It shows that the cross-classified methods do not automatically solve the identification problem; rather, they address this problem by implicitly imposing multiple constraints on the age, period, and cohort effects. These constraints not only depend on the width of the age, period, and cohort intervals but also have non-trivial implications for estimation. Because these assumptions are extremely difficult, if not impossible, to verify in empirical research, they are qualitatively no different from other constrained estimators' assumptions. The authors conclude that CCFEM and CCREM cannot and should not be used to recover the true age, period, and cohort effects.

Introduction

Demographers, sociologists, and epidemiologists have frequently attempted to separate age, period, and cohort effects because such a decomposition of data can provide important clues for the causes of trends in various outcomes including fertility, mortality, crime rates, and disease incidence. Age-Period-Cohort (APC) models are designed to estimate the independent effects of age, period, and cohort. It is well-known that classic APC models, where age, period, and cohort groups have equal interval widths, are affected by an *identification problem*. This problem arises from the exact linear dependence between the three variables: once information about any two of the three variables is available, the value for the third is completely determined, that is, Cohort = Period − Age. As a result, there exist an infinite number of possible solutions with identical fit to the data and none can be deemed the uniquely preferred solution.

Various techniques have been developed to circumvent this identification problem. Examples include the "conventional" Constrained Generalized Linear Model (Mason et al. 1973), APC Characteristic Models (O'Brien 2000), the Intrinsic Estimator (Fu 2000, Yang et al. 2004, 2008), and the Mechanism-Based Approach (Winship and Harding 2008). Recently, Yang and Land (2006, 2008) proposed a new method, cross-classified APC models[1], including Cross-Classified Fixed Effects Models (CCFEM) and Cross-Classified Random Effects Models (CCREM). They argued that the cross-classified method, using unequal interval widths for age, period, and cohort groups, breaks the exact linear dependency and thus solves the identification problem, so reliable and valid estimates for age, period, and cohort effects can be obtained. Since

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[1] The cross-classified APC models are sometimes called Hierarchical APC (HAPC) models, where age is usually included as an individual-level variable, and period and cohort are considered contextual-level factors. We do not use the name HAPC for the cross-classified APC models for the following reason: In statistics, hierarchical models assume that the unit of analysis, e.g., individuals, is nested within contextual factors such as schools and neighborhoods. However, in APC analysis, individuals are not nested in period or cohort groups; rather, period and cohort as well as age are crossed.

its introduction, this method has been adopted quickly by sociologists and demographers to address substantive issues including mortality, happiness, religious activities, and obesity (see, e.g., Masters 2012, Masters et al. 2012, Schwadel 2010, Reither et al. 2009, Yang 2008).

However, we show in this paper that although the age, period, and cohort groups of unequal interval widths used in the cross-classified APC models are not exactly linearly dependent in the relatively simple way that they are in the APC models with equal interval widths, it is still true that the independent effects of age, period, and cohort cannot be estimated without constraints. In fact, the cross-classified APC strategy implicitly uses multiple constraints to choose one set of estimates from among an infinite number of possible estimates. As a result, CCFEM and CCREM estimates can be highly biased and substantive conclusions can be misleading when any of the multiple constraints is not satisfied by the true age, period, and cohort effects. Because there is usually little or no external information to verify the validity of these constraints in empirical research, neither CCFEM nor CCREM can be used to obtain valid estimates for the true, independent effects of age, period, and cohort.

This paper contributes to the literature in three ways. First, it enriches the theoretical discussion about the identification problem in APC analysis. The current APC literature focuses on the estimation challenge in APC models with equally spaced intervals, and the methodological problems for APC models with unequally spaced intervals are yet to be fully understood. Drawing on two articles in the 1970s and 80s, we use statistical theory to demonstrate that the identification problem remains for the cross-classified APC models where age, period, and cohort groups have unequal interval widths. Second, despite the recent popularity of the cross-classified method, the validity and statistical properties of this method have not been thoroughly examined with statistical theory and simulation studies. This paper fills

this gap by identifying the multiple implicit constraints that CCFEM and CCREM impose, so that these methods are in fact simply constrained APC estimators, and illustrating the non-trivial implications of these constraints using simulations. Third, as Fu and colleagues (2011) suggested, "the important statistical issue about APC modeling is how to identify the trend that helps to resolve the real-world problem for a given APC data set" (p. 455). We address this issue by examining how the multiple constraints imposed by CCFEM and CCREM affect their ability to recover the true trends of age, period, and cohort in different circumstances. With this illustration of the implications of applying CCFEM and CCREM in empirical research, researchers who attempt to use CCFEM and/or CCREM will better understand the nature and application scope of these cross-classified APC methods.

This paper is organized as follows. To contextualize the APC method under investigation, we begin with a brief review of the identification problem in APC analysis, the cross-classified APC models including CCFEM and CCREM, as well as other constrained estimators that are intended to address this problem. The developers of CCFEM and CCREM have claimed that these cross-classified methods solve the identification problem that has haunted APC analysis for decades. However, in the theoretical exposition below, using statistical theory and mathematical derivations, we show that the identification problem remains intact and the cross-classified approach is mathematically equivalent to a constrained method that places multiple constraints on the parameter vector. These sections are fairly technical, so readers may skip them on a first reading and go directly to the "Simulation Results" section.

In the "Simulation Results" section that follows, we use simulation experiments, where the true age, period, and cohort effects are known, to assess the performance of CCFEM and CCREM under various circumstances. If the cross-classified method solves the identification

problem as its developers claim, CCFEM and CCREM estimates should be close to the true age, period, and cohort effects that we know *a priori* because we are simulating the data from known mechanisms. Yet our simulations show that CCFEM and CCREM estimates can be highly biased, very different from the true age, period, and cohort effects. The conclusion is that CCFEM and CCREM do not in fact solve the identification problem in estimating the true effects and they cannot be used to recover true age, period, and cohort effects.

The Identification Problem, Constrained APC Estimators, and Cross-Classified Fixed / Random Effects Models

To separate the independent age, period, and cohort effects, APC researchers have conventionally used an Analysis of Variance (ANOVA) model where age, period, and cohort groups have an equal interval width:

$$g(E(Y_{ij})) = \mu + \alpha_i + \beta_j + \gamma_k, \tag{1}$$

for age groups $i = 1,2,\dots,a$, periods $j = 1,2,\dots,p$, and cohorts $k = 1,2,\dots,(a+p-1)$, where $\sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{p} \beta_j = \sum_{k=1}^{a+p-1} \gamma_k = 0$. $E(Y_{ij})$ denotes the expected value of the outcome of interest Y for the ith age group in the jth period of time; g is the "link function" linking the expected value of Y_{ij} to the effects; α_i denotes the mean difference from the global mean μ associated with, or the effect of, the ith age category; β_j denotes the effect of the jth period; γ_k denotes the effect of the kth cohort. The usual ANOVA constraint applies, under which the sum of coefficients for each effect is set to zero. Note that in model (1), age, period, and cohort groups are assumed to have equal-width intervals. For example, one common practice in demographic research is to assume the interval widths for age, period, and cohort groups are all

five years; that is, for age groups, the intervals are 0-4, 5-9, …, 90-94; the period intervals are

1980-1984, 1985-1989, …, 2005-2009; and for cohorts, 1890-1894, 1895-1899, …, 2000-2004.

When the outcome Y_{ij} treated as a normal (Gaussian) random variable, $g(E(Y_{ij}))$ is

customarily just $E(Y_{ij})$, and model (1) can also be written in a generic regression fashion:

$$Y = Xb + \varepsilon, \tag{2}$$

where Y is a vector of outcomes; X is the design matrix; b denotes a parameter vector whose

elements are the effects of the age, period, and cohort groups; and ε denotes random errors with

distribution centered on zero. Then estimated age, period, and cohort effects can be obtained

using the ordinary least squares (OLS) method:

$$\hat{b} = (X^T X)^{-1} X^T Y. \tag{3}$$

However, it is well-known that the inverse of the matrix $(X^T X)^{-1}$ does not exist because

of the age-period-cohort linear dependency. In other words, the design matrix has rank one less

than full rank and thus the parameter vector b is inestimable without additional information. This

is the so-called identification problem in APC analysis: no unique set of coefficient estimates can

be obtained because an infinite number of solutions $\hat{b}$ have *identical* fits to the data.

Various methods have been developed to address the identification problem, many of

which involve imposing a constraint on the parameter vector b. For example, Mason et al. (1973)

proposed the Constrained Generalized Linear Model (CGLM), a coefficient-constrained

approach that has been considered a conventional method for APC analysis. This method places

at least one identifying restriction on the parameter vector b in equation (2). Usually the effects

of the first two age groups, periods, or cohorts are constrained to be equal. With this one

additional constraint, the APC model becomes just-identified, and unique least squares and

maximum likelihood estimators exist. The theoretical foundation of CGLM is to use external, extra information to make APC models estimable.

The CGLM approach has been criticized for relying on external or side information to specify constraints when such information often does not exist or cannot easily be verified. Also different choices of identifying constraints can produce widely different estimates of patterns of change across the age, period, and cohort categories. That is, estimates of the effect coefficients are quite sensitive to the choice of the equality coefficient constraint (Rodgers 1982a, 1982b, Glenn 2005).

Another example of constrained estimators is the Intrinsic Estimator (IE) (see Fu 2000, Yang et al. 2008). Scholars have recently recognized and discussed the merits and limitations of IE (Luo 2012, O'Brien 2011a, 2011b, Fu, Yang, and Land 2011). They show that IE implicitly assumes a specific constraint on the linear age, period, and cohort effects. This implicit constraint not only depends on the number of age, period, and cohort groups, but also is extremely difficult, if not impossible, to verify in empirical research. In this key respect, IE is no different from the CGLM method except that the CGLM constraint is explicit and does not change automatically as the number of age, period, and cohort groups changes.

Yang and Land (2006, 2008) recently proposed a new method that appears to involve no such constraints. In contrast to conventional APC methods such as CGLM and IE that use equally spaced intervals, the novelty of the cross-classified APC method is to use unequally spaced intervals for age, period, and cohort groups to break the linear dependency. Specifically, for individual-level data where each person's exact age and cohort membership and the time of data collection are known, the developers of this method proposed creating age, period, and cohort groups with different interval widths so that the exact linear dependency between such

groups disappears. For example, when the individual-level data are represented using one-year age groups, two-year periods, and five-year cohorts, one cannot tell a person's age based on knowing that person's period and cohort groups.

These authors then proposed analyzing these aggregated data treating period and cohort effects as either fixed effects or random effects, termed CCFEM and CCREM respectively. The matrix form of CCFEM and CCREM is:

$$Y = W\beta + Zu + \varepsilon, \tag{4}$$

where W and Z are the design matrices for age effects and the period and cohort effects, respectively. In CCFEM, β and u are just regression coefficients. In CCREM, u is normally distributed around a zero mean, i.e., $u \sim N(0, \sigma^2)$, so the random coefficients u are "shrunk" towards zero by the mechanism of a random effect.

The developers of CCFEM and CCREM recommended that either of these two models can be used to identify the independent age, period, and cohort components of change (Yang and Land 2006, p. 77). CCFEM and CCREM have been adopted quickly by applied researchers to analyze age, period, and cohort patterns for important substantive issues including mortality, religious activity, verbal test scores, and obesity (e.g., see Masters 2012, Masters et al. 2012, Sarma et al. 2012, 2011, Schwadel 2010, Reither et al. 2009, Yang 2008). These methods have been embraced by researchers in many fields because it appears that, unlike CGLM or IE, CCFEM and CCREM do not require subjective assumptions to find a set of uniquely preferred estimates of age, period, and cohort effects.

However, in the following sections, we show that CCFEM and CCREM do in fact impose multiple constraints on the parameter vector (β, u). This feature of CCFEM and CCREM is no different from the constraints used in methods such as CGLM and IE. Specifically, we will

identify the constraint that CCFEM and CCREM impose and then use simulation experiments to illustrate the implications of applying this method to estimate the true age, period, and cohort patterns in empirical research.

The Multiple Block Constraints (MBCs) that CCFEM and CCREM Impose

Although most of the APC literature centers on the estimation problem inherent in APC models that use equally spaced intervals, problems with unequally spaced APC intervals have also been discussed since the 1970s. Unfortunately, this literature has largely been ignored, resulting in a misunderstanding that the identification problem vanishes when age, period, and cohort groups have different interval widths. To clarify this misunderstanding, we first review two key articles that address identification with unequally spaced APC intervals and then show how the cross-classified approach chooses a set of unique solutions to APC models (2) and (4) by using multiple constraints.

Fienberg and Mason (1979) were the first to explicate the identification problem in APC models that use unequally spaced intervals, showing that as with equally spaced intervals, only nonlinear components of the age, period, and cohort effects are estimable. Specifically, using an example where the width of period intervals is twice that of age intervals, Fienberg and Mason demonstrated that "[n]ot only does this identification problem remain, but there is an additional age-cohort dependency." (p. 38) They found that as in the usual case in which APC groupings have the same interval width, imposing a restriction on the linear effects is necessary to separate the effects of age, period, and cohort. But, unlike the simpler case with equally spaced APC intervals, "[n]ot just one but two restrictions are necessary to identify all the effects" and "the two restrictions cannot be placed arbitrarily." (p. 39) That is, the two constraints have to be chosen carefully because many sets of two constraints will not identify the model.

Holford (1983, 2006) used a matrix approach for unequal-width grouping to show that none of the linear APC trends is estimable without additional constraints. Moreover, using unequally spaced intervals leads to an identification problem not only for a model where age, period, and cohort effects are all of interest but also for a subset model that includes only age and cohort effects. In other words, the method of differential groupings retains the old difficulty in identifying APC model (1) and creates a new one as well.

This literature implies that CCFEM and CCREM must impose two or more constraints to make it possible to separate age, period, and cohort effects. In general, suppose that the original individual-level data that we have are collected every year and each person's exact age and birth cohort are known. To apply CCFEM and CCREM, we reorganize the data so that there are a age groups, p periods, and c cohorts, and the age, period, and cohort intervals have widths of l, m and n years, respectively. To estimate independent age, period, and cohort effects, CCFEM and CCREM assume that the l year-specific age effects within each of the a aggregated age groups are equal, the m year-specific period effects within each of the p period groups are equal, and the n year-specific cohort effects in each of the c cohort groups are also equal. That is,

$$\alpha_{i1} = \alpha_{i2} = \cdots = \alpha_{il}, \tag{5}$$

$$\beta_{j1} = \beta_{j2} = \cdots = \beta_{jm}, \tag{6}$$

and

$$\gamma_{k1} = \gamma_{k2} = \cdots = \gamma_{kn}, \tag{7}$$

for age groups $i = 1,2,\ldots,a$, periods $j = 1,2,\ldots,p$, and cohorts $k = 1,2,\ldots,c$. Thus the assumption implicit in CCFEM and CCREM is that the true age, period, and cohort effects <u>must</u> satisfy equations (5) through (7). We term these constraints the cross-classified APC models' "Multiple Block Constraints" (MBCs). If, in fact, the true age, period, and cohort effects do not

satisfy *any one* of these equations, then the MBCs implicit in CCFEM or CCREM are incorrect. As we will see, the effects of failure to satisfy these implicit MBCs can be large and contrary to intuition.

For ease of illustration and without loss of generality, suppose we have three age categories, three periods, and thus five cohorts in the original individual-level data so that $a = 3, p = 3$ and $c = 5$. For each age-by-period combination, there is one observation. The design matrix X for CCFEM using the sum-to-zero constraint is given in Table 1. With the last group omitted for each variable, the design matrix X consists of one column for the intercept, $a - 1 = 2$ columns for the age dummy variables, $p - 1 = 2$ columns for period, and $c - 1 = 4$ columns for cohorts. If more age, period, and cohort categories are included, Table 1 can be expanded, with each new observation corresponding to an additional row in the design matrix. In the case where age or period is treated as a continuous variable, simply replace the columns for age or period effects with the values of linear and non-linear age or period terms for each observation based on the model specification.

[Table 1 is about here]

The design matrix in Table 1 has rank one less than full rank because of the linear dependency between age, period, and cohort. That is, the number of linearly independent columns is one less than the number of columns in the design matrix. One way to make model (1) estimable, i.e., to modify the design matrix in Table 1 so that it has full rank, is to constrain two adjacent cohorts to have equal effects; that is, to assume $\gamma_1 = \gamma_2$ and $\gamma_3 = \gamma_4$. Let $\gamma_1' = \gamma_1 = \gamma_2$ and $\gamma_2' = \gamma_3 = \gamma_4$. Then the design matrix of APC model (1) assuming equal effects for two adjacent cohorts has full rank and can be rewritten as in Table 2.

[Table 2 is about here]

The design matrix in Table 2 is identical to that for CCFEM using single-year age groups, single-year periods, and two-year cohorts, that is, $l = 1, m = 1$ and $n = 2$ so that there are three one-year age groups, three one-year periods, and three two-year cohorts after grouping. In other words, applying CCFEM to this example using two-year cohort is mathematically equivalent to assuming that the two one-year cohorts within each of the two-year cohort groups have the same effects. When age or period is grouped into multi-year categories, the CCFEM method places further constraints on the age or period effects in addition to the constraints on cohort effects, so that the single-year age or period groups within each multi-year group have equal effects.

A similar exposition applies to CCREM. The design matrices for the fixed and random effects, W and Z, using equally spaced intervals for the data above are given in Table 3. Specifically, W has one column for the intercept and $a - 1$ columns for the fixed age effects. Z has $p + c$ columns for the random period and cohort effects. Like the CCFEM design matrix in Table 1, the CCREM design matrix in Table 3 has rank one less than full rank. One way to make the design matrix have full rank (i.e., to address the identification problem) is to constrain two adjacent cohorts to have equal effects, that is, to constrain $\gamma_1 = \gamma_2$ and $\gamma_3 = \gamma_4$. Let $\gamma_1' = \gamma_1 = \gamma_2$ and $\gamma_2' = \gamma_3 = \gamma_4$. Then the CCREM design matrix has full rank and can be rewritten as in Table 4. The design matrix in Table 4 is identical to that for CCREM using single-year age groups, single-year periods, and two-year cohorts. In other words, applying CCREM to the data set above using two-year cohorts is mathematically equivalent to assuming that the two one-year cohorts within each of the two-year cohort groups have the same effects.

[Table 3 is about here]

[Table 4 is about here]

More importantly, the exposition above indicates that the MBCs implicit in the CCFEM and CCREM methods also depend on the design matrix, i.e., on the number of age, period, and cohort groups, when their interval widths change. For example, if we use a three-year interval for cohort group, so we now have three one-year age groups, three one-year periods, and two three-year cohorts, then following the same derivation used above, the MBCs implied by CCFEM and CCREM are $\gamma_1 = \gamma_2 = \gamma_3$ and $\gamma_4 = \gamma_5$. Compared to the constraints for the case of three two-year cohorts, increasing the interval width for cohort (or alternatively, decreasing the number of cohorts) for the same set of individual-level data changes the constraints on cohort effects so that true effects satisfying CCFEM and CCREM's MBCs with two-year cohorts no longer satisfy the constraints when the width of cohort intervals is increased by one year (or alternatively, when the number of cohorts is reduced by one). Readers can verify that increasing or decreasing the width of age or period intervals also alterS the MBCs that CCFEM and CCREM implicitly assume.

These examples demonstrate that not only do CCFEM and CCREM rely on a constraint like CGLM does, but unlike CGLM, where the constraint (e.g., equal age effects for the first two age groups) is explicit and can be rationalized by a theoretical account or side information, the MBCs implied by CCFEM and CCREM are implicit and vary depending on the width (or equivalently the number) of the age, period, and cohort groups. Although the MBCs that CCFEM and CCREM use to circumvent the identification problem have not been recognized in the sociological and demographic literature, in fact, as we will show with simulated data, these MBCs can have large and non-intuitive consequences for estimating age, period, and cohort patterns. This instability of CCFEM and CCREM's implicit MBCs further limits the application of the two methods for analyzing data in combination with lack of *a priori* or side information about the age, period, and cohort effects.

In sum, CCFEM or CCREM can be viewed as a constrained estimator that places *multiple* equality constraint on the coefficient vector (β, u). Since the constraint assumption strongly affects estimation results, a constrained method produces good estimates only when its assumption approximates the true structure of the data under investigation. For any coefficient-constraint approach such as CCFEM and CCREM, "the choice of constraint is the crucial determinant of the accuracy in the estimated age, period, and cohort effects" (Kupper et al. 1985, p. 822). It follows that only when the effects of age, period, and cohort in each of the multi-year APC categories are equal can CCFEM and CCREM yield accurate estimates. If, in fact, the true age, period, and cohort trends do not satisfy this assumption, then the CCFEM and CCREM estimates will be incorrect and may be highly distorted. However, APC researchers usually have no *a priori* knowledge about the relationship among age, period, and cohort effects. Therefore, unfortunately, researchers cannot know what datasets CCFEM and CCREM can be applied to, so in this respect these methods are no better than conventional constrained methods like CGLM and IE. To the extent that CCFEM and CCREM impose more than one constraint on the effects of age, period, and cohort respectively, it means that CCFEM and CCREM require substantially more side information than CGLM to produce estimates. Therefore, it is more challenging to verify the MBCs of CCFEM and CCREM than for the conventional CGLM, which requires only one equality constraint to obtain unique estimates for age, period, and cohort effects.

Simulation Results: Implications of the Multiple Block Constraints (MBCs) and Application Scope of CCFEM and CCREM in Empirical Research

While the preceding discussion about the implicit Multiple Block Constraints (MBCs) that CCFEM and CCREM impose is fairly technical, simulation results in this section offer a straightforward illustration of the technical exposition. We first use a set of simple simulations to

illustrate the implications of the MBCs that CCFEM and CCREM impose. We then use several types of simulated data that embody specific social theories to examine how these MBCs affect CCFEM's and CCREM's ability to recover the true age, period, and cohort trends in social science research[2].

To illustrate the implications of the MBCs assumed by CCFEM and CCREM, we simulate normally distributed micro data sets (i.e., individual-level data) as follows. For persons at age i in period j, the mean response is $0 + \alpha_i + \beta_j + \gamma_{ij}$ and the standard deviation of error ε equals 1. The number of age and period groups is fixed at three each in the micro data, so there are three age groups, three periods, and five cohorts at the individual level. We consider three sets of true α_i, β_j, and γ_{ij} as shown in Table 5. For each selection of true α_i, β_j, and γ_{ij}, we simulated 10,000 individuals. We then repeat the simulation process 100 times by drawing random errors to obtain 100 such data sets, each of which has 10,000 observations. To minimize the impact of an unbalanced study design on parameter estimation for random effects models, the number of observations in each combination of age and period is equal.

[Table 5 about here]

To apply the cross-classified APC models to the simulated data, we created one-year age groups, one-year periods and two-year cohorts so that the design matrix using such groupings has full rank. We then include the three aggregated age, period, and cohort variables as fixed effects in CCFEM. CCREM is implemented using fixed age effects and random period and cohort effects.

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[2] The researchers who created CCFEM and CCREM claimed to use empirical data, where the true effects are unknown, to validate the two methods (see Yang and Land 2006, 2008, Yang et al. 2008). However, it is logically impossible to assess an estimator's ability to estimate true effects when the true effects are unknown. Therefore, cross-model comparisons using empirical data are not an appropriate method to validate CCFEM or CCREM.

Note that for the first data set, the relationship between the age, period, and cohort effects satisfies equations (5) through (7), the MBCs implicit in CCFEM and CCREM; as shown in Table 5, for the first set of true α_i, β_j, and γ_{ij}, $\gamma_1 = \gamma_2, \gamma_3 = \gamma_4$. In contrast, the MBCs for cohort effects in the cross-classified APC models do not hold for the second and third data sets.

Table 6 presents the CCFEM and CCREM estimates, averaged over the 100 simulated data sets, for the three sets of true α_i, β_j, and γ_{ij}, along with the true effects. The biases of CCFEM and CCREM are estimated by the difference between the truth and the estimates averaged over the 100 simulated data sets. For the first data set, both CCFEM and CCREM yield good estimates for the true α_i, β_j, and γ_{ij} because the true age, period, and cohort effects satisfy equations (5) through (7), the implicit MBCs imposed by the two methods. In contrast, for the second and third set of true α_i, β_j, and γ_{ij}, CCFEM and CCREM return highly biased estimates, very different from the true effects, because these true effects do not satisfy the MBCs implicit in CCFEM and CCREM. For the second data set, the estimated period effects, averaged over the 100 simulated data sets, show a downward trend when the true trend is upward. Similarly, for the third data set, when the true trend in age effects is upward, the CCFEM and CCREM estimates suggest that the outcome of interest does not vary across age groups. For the second and third data sets, the cohort effect estimates are effectively zero although the true cohort effects show increasing and decreasing trends, respectively.

[Table 6 about here]

Given that CCFEM and CCREM impose MBCs on the underlying age, period, and cohort trends, the question becomes how these MBCs affect the ability of these cross-classified methods to produce unbiased and consistent estimates of the true patterns. Thus the second objective of this simulation study is to examine whether CCFEM and CCREM are able to

capture the temporal patterns associated with age, period, and cohort. Specifically, using several types of simulated data, we examine whether CCFEM and CCREM yield good if not unbiased estimates that recover age, period, and cohort patterns that may be observed in practice. For a good statistical method, one should expect that the resulting estimates are at least close to the values specified in the simulation function if they are not unbiased. It follows that if CCFEM or CCREM is a useful method for APC researchers to obtain reliable estimates of true trends, CCFEM and CCREM estimates should be close to the true effects of age, period, and cohort that are known from the data-generating mechanisms.

We simulate data from four scenarios that embody specific social theories. If these theories reflect social realities, this is what the data would look like, so it is essential to estimate the effects predicted by these theories without distortion. The first scenario is simulated to represent the common observation that while body weight tends to increase with age and peak around age 60, obesity rates have increased alarmingly in recent past decades in the United States (Ogden 2006, Flegal et al. 2002, Mokdad et al. 2001). At the same time, epidemiological theories suggest that rising obesity may be attributed to one's *in utero* environment. For example, the fetal over-nutrition theory posits that increasing *in utero* exposure to maternal obesity may lead to an inter-generational increase in offspring obesity (Cole et al. 2008, Gillman 2004). This theoretical argument suggests that the obesity epidemic may manifest as cohort effects, as each successive cohort is at higher risk for obesity. On the other hand, the epidemiological, sociological, and demographic literature has also suggested that age, period, or cohort effects may not all exist (Keyes et al. 2009, Winship and Harding 2008, Fabio et al. 2006, Preston and Wang 2006, Raftery et al. 1995, Ryder 1965). Accordingly, the other three scenarios use

simulated data that approximate the likely empirical situation if age, period, or cohort has little impact on the outcome of interest.

Specifically, to simulate individual-level data, we fix the number of age groups at 20 and periods at 20 in all of these simulations with little loss of generality. For each age and period combination, we simulated 25 individuals, and total sample size for each data set is thus 10,000. We then generate 100 such individual-level data sets, each of which has 10,000 observations, from each of the following four scenarios with means $E(Y_{ij})$ as a function of age, period, and cohort:

$$0.3 \cdot age_i - 0.01 \cdot age_i^2 - 0.04 \cdot period_j + 0.02 \cdot period_j^2 + 0.35 \cdot cohort_{ij} - 0.0015 \cdot cohort_{ij}^2 \quad (8)$$

$$0.04 \cdot period_j + 0.02 \cdot period_j^2 + 0.35 \cdot cohort_{ij} - 0.0015 \cdot cohort_{ij}^2 \quad (9)$$

$$0.3 \cdot age_i - 0.01 \cdot age_i^2 + 0.35 \cdot cohort_{ij} - 0.0015 \cdot cohort_{ij}^2 \quad (10)$$

$$0.3 \cdot age_i - 0.01 \cdot age_i^2 - 0.04 \cdot period_j + 0.02 \cdot period_j^2 \quad (11)$$

In each scenario, the data are normally distributed with these means and the error standard deviation $\sigma = 1$, For instance, in Scenario 1 with mean function (8), the body weight outcomes for people in age i in period j are normally distributed with mean $0.3 \cdot age_i - 0.01 \cdot age_i^2 - 0.04 \cdot period_j + 0.02 \cdot period_j^2 + 0.35 \cdot cohort_{ij} - 0.0015 \cdot cohort_{ij}^2$ and the standard deviation σ for people of age i in period j is 1. In Scenarios 2 through 4, with mean functions (9), (10), and (11) respectively, one of the age, period, and cohort effects is not present while the effects of the other two variables are the same as in function (8).

To apply the cross-classified APC models to the four scenarios, we create one-year age groups, two-year periods and five-year cohorts as Yang and Land (2008) did, so the design matrix using such groupings has full rank. To examine whether the estimates of the cross-classified APC methods depend on the width of age, period, and cohort groups, we use two-year

intervals for cohorts and compared the results with the estimates obtained using five-year cohorts for the same data.

Figure 1 and Figure 2 present the CCFEM estimates and CCREM estimates, respectively, using different interval widths for cohorts for the simulated data from the four scenarios, along with the true age, period, and cohort effects specified in functions (8) to (11). The CCFEM and CCREM estimates shown in these figures are averaged over 100 data sets, so an estimate of the bias of these methods is the difference between the estimates and true effects. Figures 1 and 2 show that the CCFEM and CCREM estimates suffer from two types of errors: errors in estimating the overall trends in age, period, and cohort categories, and errors in estimating the trends within each aggregate age, period, and cohort groups. For the overall age, period, and cohort trends, the CCFEM and CCREM coefficient estimates obtained using different widths for cohorts, averaged over 100 data sets, are largely away from true effects for all the models because for all four scenarios, the MBCs that CCFEM and CCREM assume are not satisfied. For example, in Scenario 1 with substantial age, period, and cohort patterns in both figures, CCFEM and CCREM results indicate an age trend contrary to the true trend for the 15th and older age groups and opposite period trends for the 10th and previous periods. The conclusion about cohort patterns based on CCFEM and CCREM results are qualitatively incorrect: although the true cohort effects show a strong positive trend, CCFEM estimates using five-year cohorts and both CCREM estimates suggest a slight bell-shaped trend among these cohorts. Similarly, in Scenario 2, 3, and 4, when there is no age, period, or cohort effect respectively in the data generating mechanisms, the CCFEM and CCREM estimates suggest a substantially positive or negative age, period, or cohort trend along with inaccurate estimates for the other two variables.

[Figure 1 about here]

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The second type of error occurs in estimating the effects within the multi-year groups. For example, in Scenario 3 with a dominant true linear trend across cohort groups in both figures, the cross-classified models using two-year cohorts assume that two adjacent cohorts have the same effects, and the models using five-year cohorts assume that the five one-year cohorts within each aggregated five-year cohort groups have the same effects. This is not surprising because detailed information is lost in lumping single-year group to create multi-year age, period, and cohort categories.

Moreover, Figures 1 and 2 show that CCFEM and CCREM estimates depend on the widths of age, period, and cohort intervals. For example, in Figure 1 in Scenarios 1 and 2 with a positive cohort trend, while CCFEM using five-year cohorts indicates a slight bell-shaped pattern, the same method using two-year intervals for cohorts suggests a negative cohort pattern. As another example, in Figure 2 in Scenario 4 with no cohort effects, CCREM estimates using two-year cohorts indicate a much steeper negative trend across cohort groups than the estimates from the same method using five-year intervals for cohorts. Interested readers can verify that changing the interval width for age and/or period groups can also alter the estimated age, period, and/or cohort trends. Clearly, the cross-classified approach not only retains the identification problem inherent in APC model (1) but also creates further difficulties for understanding the variation within each aggregated age, period, and groups. In sum, in specific applications, CCFEM and CCREM must be assumed to be biased, resulting in potentially misleading conclusions about the true age, period, and cohort effects unless proven otherwise.

Conclusion and Discussion

This paper focuses on the Cross-Classified Fixed Effects Model (CCFEM) and Cross-Classified Random Effects Model (CCREM), two statistical methods intended to separate the independent effects of age, period, and cohort on an outcome of interest. We have discussed the nature and application scope of CCFEM and CCREM theoretically and illustrated them with simulated data. CCFEM and CCREM assume Multiple Block Constraints (MBCs) on the age, period, and cohort effects. These MBCs not only depend on the widths of age, period, and cohort intervals, but also are extremely difficult, if not impossible, to verify in empirical research. This feature of CCFEM and CCREM is no different from the constraint assumed in CGLM except that the CGLM is explicit and usually imposes just one constraint. The simulations imply that CCFEM and CCREM are not unbiased estimators of the true age, period, and cohort effects. Therefore, for social scientists whose goal is to reveal the true, simultaneously independent effects of age, period, and cohort, CCFEM and CCREM's strategy of circumventing the identification problem does not, in fact, reliably produce estimates with desirable properties. It can yield highly biased estimates and is thus potentially misleading.

APC studies that apply CCFEM and/or CCREM often report coefficient estimates accompanied by their estimated standard errors and researchers use them to make statistical inference about age, period, and cohort effects; such estimates of standard errors are available automatically as part of the program output from statistical packages. The developers of the cross-classified method suggested that CCREM should be preferred to CCFEM if CCREM has smaller standard error estimates. However, we do not encourage using the estimated standard errors to compare the performance of different APC models or to make statistical inference because conclusions based on such standard errors ignore the potentially gross bias in coefficient estimation, an estimation bias that is "the primary reason why patterns in estimated age, period,

and cohort effects vary so much as a function of the additional linear constraint" (Kupper et al. 1985, p. 822).

It is worth emphasizing that the cross-classified method chooses to throw away information by "lumping" individual data into aggregated groups to break the linear dependence among age, period, and cohort. This is surprising because the mathematical and logical arguments that we have presented about identifiability depend in no way upon the degree of grouping. That is, neither finer nor coarser grouping itself can resolve the identification problem of these models. Although the creators of the method claim that unequally spaced grouping has solved the problem, we have shown that the true source of the "solution" is the assumption that the effects are identical within each of the aggregated age, period, and cohort groups.

The idea of using individual data to achieve identifiability is not new in the APC literature. For example, Boyle and Robertson (1987) argued that "[a] solution to this [identification] problem lies in using non-aggregated data, i.e., data available in the form of individual records." (p. 733) By constructing non-overlapping cohorts using individual records, Boyle and colleagues (1983, 1987) and Robertson and Boyle (1986) proposed an APC model which, they claim assumed, no arbitrary constraints. However, this individual records approach does not in fact solve the problem (see, e.g., Clayton and Schifflers 1987, Tango 1988, Osmond and Gardner 1989), and Robertson and Boyle (1998) later noted that the assumption imposed by their method is invalid (p. 1311).

In some expositions of CCFEM and CCREM, the developers of the cross-classified method included both linear and quadratic age terms in the cross-classified models and claimed that "it is clear that the underidentification problem of the classical APC accounting model has been resolved by the specification of the quadratic function for the age effects" (Yang and Land

2008, p. 84). However, adding a quadratic age term alone does not break the *linear* dependency noted above or solve the identification problem because the linear age, period, and cohort effects are exactly related in APC models with unequal interval widths, as they are in the model with equal interval widths.

What should researchers do if they want to investigate age, period, and cohort patterns? While several alternative methods have been developed, some of which are more theoretically driven by taking external or side information into account[3] while others are purely statistical,[4] it is clear from the above discussion and the current APC literature that there has been no satisfactory general solution of the identification problem. Indeed, the use of the APC model (1), regardless of estimation technique, precludes valid estimation as well as meaningful interpretations of the linear components of age, period, and cohort effects (see, e.g., Holford 1983, Kupper et al. 1985). The simple fact that age, period, and cohort are perfectly related ensures that any modeling efforts directed at search for a general solution of independent, additive age, period, and cohort effects is futile. As Clayton and Schifflers (1987) pointed out, "[n]o sophiciation of method can create information where that information is lacking" (p. 479). Therefore, we should encourage the development of APC models that are informed by social theories and thus different from model (1) in basic structure.

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[3] For example, O'Brien (2000) developed a method called "Age-Period-Cohort Characteristic Models," in which one of the three APC variables (typically cohort) is replaced by a single mechanism variable. Similarly, Winship and Harding (2008) proposed a mechanism-based approach to identifying APC models by specifying the mechanisms through which age, period, and cohort affect the outcome of interest.

[4] For example, Fu (2000) and Yang et al. (2008) proposed a constrained method called the "Intrinsic Estimator." See Luo (2012) and O'Brien (2011) for a detailed discussion of the nature and properties of this technique.

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Tables and Figure

Table 1. Design Matrix of CCFEM for Equally Spaced Age, Period, and Cohort Groups.

Intercept	Age		Period		Cohort			
	α_1	α_2	β_1	β_2	γ_1	γ_2	γ_3	γ_4
1	1	0	1	0	0	0	1	0
1	1	0	0	1	0	0	0	1
1	1	0	-1	-1	-1	-1	-1	-1
1	0	1	1	0	0	1	0	0
1	0	1	0	1	0	0	1	0
1	0	1	-1	-1	0	0	0	1
1	-1	-1	1	0	1	0	0	0
1	-1	-1	0	1	0	1	0	0
1	-1	-1	-1	-1	0	0	1	0

Table 2. Design Matrix of CCFEM for Equally Spaced Age, Period, and Cohort Groups by Constraining Two Adjacent Cohorts to Be Equal.

Intercept	Age		Period		Cohort	
	α_1	α_2	β_1	β_2	γ'_1	γ'_2
1	1	0	1	0	0	1
1	1	0	0	1	0	1
1	1	0	-1	-1	-1	-1
1	0	1	1	0	1	0
1	0	1	0	1	0	1
1	0	1	-1	-1	0	1
1	-1	-1	1	0	1	0
1	-1	-1	0	1	1	0
1	-1	-1	-1	-1	0	1

Table 3. Design Matrix of CCREM for Equally Spaced Age, Period, and Cohort Groups.

| | Fixed Effects | | | Random Effects | | | | | | | |
| | Age | | Period | | | Cohort | | | | |
Intercept	α_1	α_2	β_1	β_2	β_3	γ_1	γ_2	γ_3	γ_4	γ_5
1	1	0	1	0	-1	0	0	1	0	-1
1	1	0	0	1	-1	0	0	0	1	-1
1	1	0	-1	-1	2	-1	-1	-1	-1	4
1	0	1	1	0	-1	0	1	0	0	-1
1	0	1	0	1	-1	0	0	1	0	-1
1	0	1	-1	-1	2	0	0	0	1	-1
1	-1	-1	1	0	-1	1	0	0	0	-1
1	-1	-1	0	1	-1	0	1	0	0	-1
1	-1	-1	-1	-1	2	0	0	1	0	-1

Note: This table is no different from that for CCFEM in Table 1 expect that the random effects part of this CCREM design matrix includes columns β_3 and γ_5 that were omitted from Table 1.

Table 4. Design Matrix of CCREM for Equally Spaced Age, Period, and Cohort Groups by Constraining Two Adjacent Cohorts to Be Equal.

| | Fixed Effects | | | Random Effects | | | | | |
| Intercept | Age | | Period | | | Cohort | | | |
	α_1	α_2	β_1	β_2	β_3	γ'_1	γ'_2	γ'_3	
1	1	0	1	0	-1	0	1	-1	
1	1	0	0	1	-1	0	1	-1	
1	1	0	-1	-1	2	-1	-1	2	
1	0	1	1	0	-1	1	0	-1	
1	0	1	0	1	-1	0	1	-1	
1	0	1	-1	-1	2	0	1	-1	
1	-1	-1	1	0	-1	1	0	-1	
1	-1	-1	0	1	-1	1	0	-1	
1	-1	-1	-1	-1	2	0	1	-1	

Note: This table is no different from that for CCFEM in Table 2 expect that the random effects part of this CCREM design matrix includes columns β_3 and γ'_3 that were omitted from Table 2.

Table 5. Examples of True Age, Period, and Cohort Effects.

Dataset	Age			Period			Cohort				
	α_1	α_2	α_3	β_1	β_2	β_3	γ_1	γ_2	γ_3	γ_4	γ_5
NO. 1	-1	0	1	-1	0	1	-1.5	-1.5	0	0	1.5
NO. 2	-1	0	1	-1	0	1	-3	-1.5	0	1.5	3
NO. 3	-1	0	1	-1	0	1	2	1	0	-1	-2

Table 6. Simulation Results: CCFEM and CCREM estimates for the three data sets in Table 5.

		Dataset 1			Dataset 2			Dataset 3		
		Truth	CCFEM	CCREM	Truth	CCFEM	CCREM	Truth	CCFEM	CCREM
Age	1	-1	-1.002	-0.999	-1	-2.500	-2.499	-1	-0.002	0.001
	2	0	0.001	-0.001	0	0.001	-0.002	0	0.000	-0.002
	3	1	1.001	1.000	1	2.499	2.501	1	0.002	0.000
Period	1	-1	-1.002	-0.999	-1	0.498	0.498	-1	-1.998	-2.002
	2	0	0.000	-0.001	0	0.000	0.000	0	0.000	0.002
	3	1	1.003	1.000	1	-0.498	-0.499	1	1.998	2.000
Cohort	1	-1.5	-1.503	-1.500	-3	-0.001	-0.001	2	0.004	0.000
	2	-1.5	-1.503	-1.500	-1.5	-0.001	-0.001	1	0.004	0.000
	3	0	-0.001	-0.001	0	0.000	-0.001	0	-0.001	0.002
	4	0	-0.001	-0.001	1.5	0.000	-0.001	-1	-0.001	0.002
	5	1.5	1.504	1.501	3	0.002	0.002	-2	-0.004	-0.002

Figure 1. Simulation Results: CCFEM estimates using two- and five-year cohort for data from four scenarios.
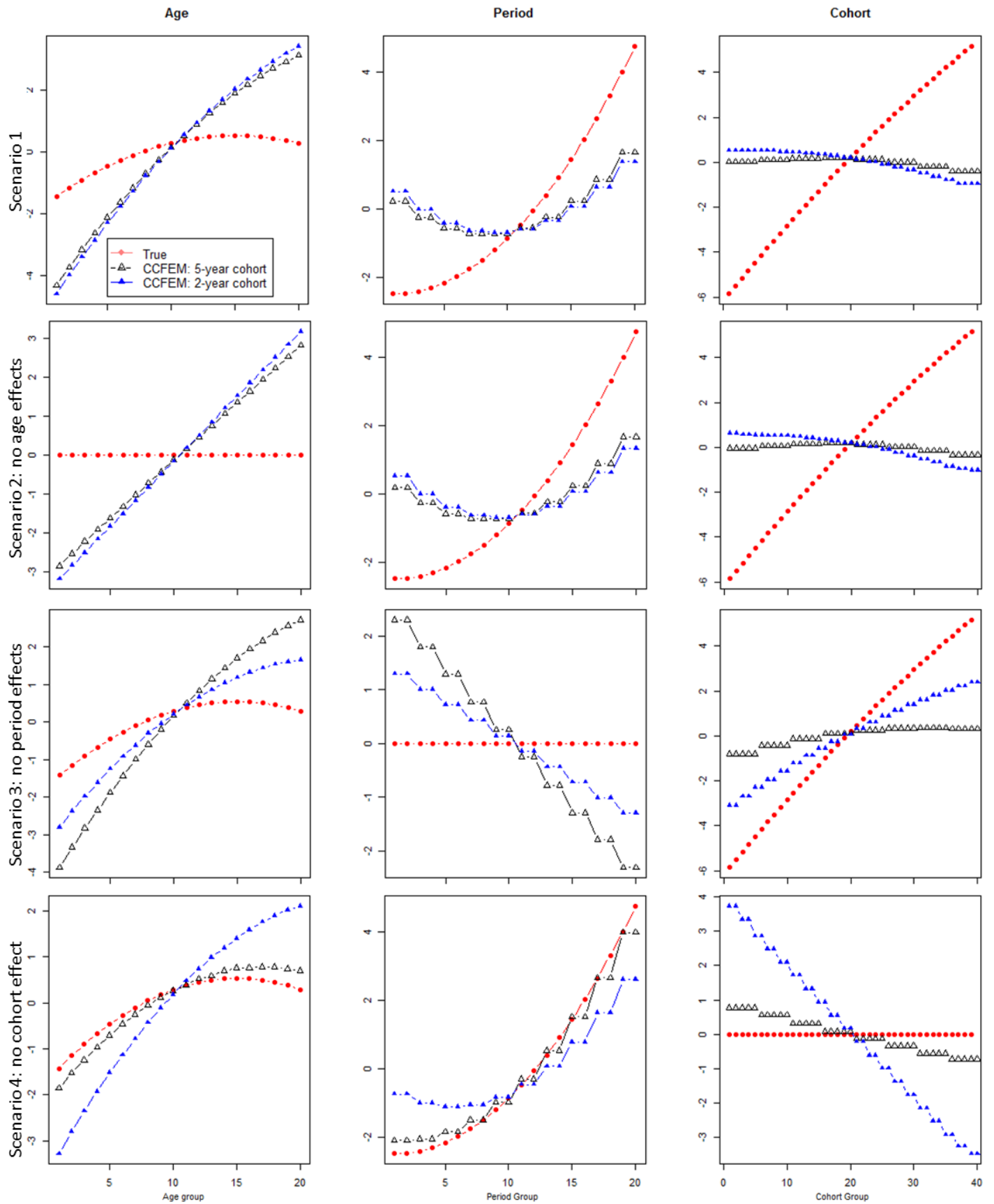
Figure 2. Simulation Results: CCREM estimates using two- and five-year cohort for data from four scenarios.