Demography, Growth, and Inequality*

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Abstract

We extend the single-sector endogenous growth model to allow for a general demographic structure. The model shows that due to the "generational turnover term," the equilibrium growth rate is less than that of a representative agent model. We find the local dynamics about the balanced growth path (bgp) to be unstable, implying that the bgp is the only viable equilibrium. Using numerical simulations, we show that the economic consequences of a change in the population growth rate differ substantially, depending on the source of the demographic change. Finally, we analyze the relationship between changes in the demographic structure and what we call the "natural rate of wealth inequality".

Keywords: Demographic change, endogenous growth, wealth inequality.

JEL Codes: J11, O41, D63

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1. Introduction

During the last half century, the demographic structure of all developed nations changed dramatically. Both fertility rates and mortality rates declined and are predicted to further do so for years to come. At an individual level, a decrease in mortality implies a longer lifespan, while at an aggregate level it means that the relative share of the elderly in society increases. Similarly, a decline in fertility implies that the inflow of young individuals into society is decreasing. Naturally, the changing demographic structure has strong implications for individual, as well as aggregate, economic outcomes particularly over the long term. Despite this, demographic structural aspects play a surprisingly small role in most macro models of economic growth.[1]

Demographic features were first introduced into macroeconomic models in the form of the overlapping generations model pioneered by Samuelson (1958) and Diamond (1965) [SD]. In its canonical form, an individual lives for 2 periods, period 1 when he/she is young and period 2 when he/she is old.[2] Some years later, a more probabilistic treatment of demographic factors was introduced via the continuous-time overlapping generations model of Blanchard (1985), Buiter (1988), and Weil (1989) [BBW]. In this set-up individuals face a constant probability of death over their lifespan and at each instant of time a new cohort is added to the economy.[3]

More recently, the Blanchard-Buiter-Weil framework has been extended to allow for a more realistic representation of individual and aggregate demographic structures. Bommier and Lee (2003) and d'Albis (2007) employ very general survival functions, while Boucekkine *et al.* (2002), Heijdra and Romp (2008), and Mierau and Turnovsky (2011) introduce empirically plausible demographic structures to study a range of theoretical and empirical issues. Although most of the contributions have dealt with exogenous growth, recently d'Albis and Augeraud-Véron (2009), Heijdra and Mierau (2012), and Bruce and Turnovsky (2011) have embedded the rectangular, the Boucekkine *et al.* (2002), and the de Moivre (1725) mortality functions, respectively, into the single-

[1] Consulting any of the leading textbooks on modern economic growth theory reveals that the representative agent model remains the dominant paradigm; see, for instance Acemoglu (2009) for the most recent and comprehensive treatment.
[2] The SD model has been used and extended extensively to allow for a variety of features (de la Croix and Michel, 2002) or many more periods (Auerbach and Kotlikoff, 1987).
[3] An important feature of the BBW model is the existence of perfect annuity markets which was first introduced by Yaari (1965) in the context of a partial equilibrium life-cycle model.

sector endogenous growth framework of Romer (1986).[4]

In this paper, we further develop the relationship between the demographic and economic growth. In particular, we construct a single-sector endogenous growth model with a rich demographic structure to study the impact of changes in the population growth rate on the economic growth rate and on the level of wealth inequality. In this respect we advance the work of d'Albis and Augeraud-Véron, Heijdra and Mierau, and Bruce and Turnovsky. Notably, in contrast to previous contributions, we do not rely on a specific parameterized survival function for our key analytical results. Hence, we are able to extend a number of earlier propositions to a general demographic composition. In addition, we are able to provide a complete characterization of the local dynamics in the neighborhood of the balanced growth path, establishing that it is in fact the only viable and sustainable equilibrium. Finally, we conduct numerical analysis simulations, which provide insights into the relation between the population growth rate and the economic growth rate, and a novel analysis of the relationship between the demographic structure and wealth inequality.

The economy we consider is populated by overlapping generations of individuals that differ only with respect to their age. An important feature of the model is that as agents age, their probability of death increases. The production side of the model consists of many individual firms that exert an investment externality on each other so that, in the aggregate, the equilibrium sustains endogenous growth in the sense of Romer (1986). For the theoretical part of the analysis we are able to establish several propositions relating to the relationship between the demographic and the macroeconomic structure.[5]

First, we show that the demographic structure imposes a negative drag on the growth rate of aggregate consumption, through the "generational turnover term". This refers to the reduction in aggregate consumption due to the addition of newborn agents having no accumulated assets,

[4] We may note that the de Moivre function, which is a special case of the Gompertz (1825) function, traditionally employed by demographers, offers the pedagogic advantage that it embeds the SD and BBW survival functions as two polar cases. However, it does not fit the data as well as does the Boucekkine *et al.* function. The "rectangular" mortality function has the characteristic that the agent survives with probability one for a fixed period, at which time he/she dies. It is essentially the assumption made in the Samuelson-Diamond model, except the length of life is potentially variable.

[5] In developing our model we have tried to work within the "small-model economics" tradition, this implies that our model abstracts from numerous real- life features such as a pension system, retirement and uncertainty. These abstractions, however, allow us to highlight very clearly the "transmission mechanisms" that are driving the results of the model (Turnovsky, 2011).

together with the departure of agents with accumulated lifetime assets. Second, we establish the existence of potentially two equilibrium growth rates. However, one of these is shown to be non-viable, in that it violates the transversality condition of the individual maximization problem, consistent with a similar result obtained by Bruce and Turnovsky (2011). The other is indeed a consistent, sustainable, interior, equilibrium growth rate. This finding effectively extends a related result by d'Albis and Augeraud-Véron (2009) from a rectangular demographic structure to a general one. Finally, we show that the growth rate associated with the feasible equilibrium is slower than the growth rate that would prevail in a representative agent economy, this being a consequence of the negative drag arising from the generational turnover term.

In general, the global dynamic analysis of an overlapping generations model having a realistic demographic structure is intractable. In view of this, we approximate and linearize the dynamic system around the balanced growth path. This allows us to establish, numerically, that away from steady state the system is unstable, so that the only viable equilibrium is for the economy to always be on its balanced growth path. This characteristic, associated with the standard representative agent endogenous growth model of Romer (1986), extends to the overlapping generations model having a very general demographic structure. [6]

We employ our model to perform two major numerical analyses. First we ask how a 0.5 percentage point increase in the population growth rate affects the economic growth rate? We find there to be a stark contrast between the economic consequences of a change in the population growth rate that is driven by an increase in the fertility rate, on the one hand, and a change that is driven by a decrease in the mortality rate, on the other. While the former leads to a *slight decline* in the economic growth rate, the latter leads to a *substantial increase* in the economic growth rate.

These findings, relating the population growth rate and the economic growth rate, can be best understood by referring to the empirical study of Kelly and Schmidt (1995). They summarize the difference by interpreting newborns as "resource users" with little accumulated wealth, and working adults with their accumulated capital as being "resource creators". While an increase in the birth rate

[6] In this respect we generalize previous analyses, embedding the Blanchard (1985) demographic structure into the one-sector Romer (1986) model, and conclude that the only viable equilibrium is for the economy always to be on its balanced growth path.

increases the former, a decrease in the mortality rate increases the latter. The positive relationship between a decrease in mortality and an increase in the growth rate is documented empirically by Bloom *et al.* (2007) and Lorentzen *et al.* (2008), who both show that high levels of adult mortality are associated with low levels of savings and growth. In a cross-sectional analysis, Sala-i-Martin *et al.* (2004) document a positive relationship between life-expectancy and economic growth, and a negative relationship between fertility and growth.

As a result of the contrasting impacts of fertility and mortality driven changes in the population growth rate on the economic growth rate, the empirical evidence linking the two growth rates is generally ambiguous. Indeed, while Kelly and Schmidt (1995) obtain a negligible relation between the population growth rate and the economic growth rate for the 1960s and 1970s, they find a negative effect in the 1980s. Similarly, Sala-i-Martin *et al.* (2004) find that the direction of the impact of the population growth rate on the economic growth rate is unclear. Our analysis suggests that these ambiguous empirical results can be reconciled by focusing on the sources of demographic change. That is, while a positive relationship between the population growth rate and the economic growth rate is associated with a drop in the mortality rate, a negative relationship is associated with an increase in the fertility rate.

The second numerical exercise we conduct involves the relationship between changes in the demographic structure and what we call the "natural rate of wealth inequality". This refers to the degree of wealth inequality that can be attributed purely to the fact that individuals of different ages are at different stages of their savings life-cycle. Although the application of this concept in an overlapping generations setting is novel, Atkinson (1971) provides an early discussion. In our analysis we find that an increase in the fertility rate leads to *slight increase* in the degree of wealth inequality while a decrease in the mortality rate leads to a *very substantial increase* in wealth inequality. As the key driving force behind the demographic change in the last decades has been the decline in mortality, we conclude that at least part of the recent dramatic increase in wealth inequality can be attributed to purely demographic phenomena; see Atkinson *et al.* (2011) for an overview.

The remainder of the paper is structured as follows. Section 2 introduces the model and

Section 3 discusses its macroeconomic equilibrium properties. Section 4 reports the numerical simulations, including some robustness analyses of the key results. Section 5 concludes, and the Appendix contains the technical details, including providing the detailed proofs of the propositions.

2. The Model

We consider a closed economy that is populated by overlapping generations of individuals that differ only in their age. The individuals supply a fixed amount of labor during their life time and must decide how much to consume now and how much to save for later consumption. The production sector comprises many individual firms that exert productive (investment) externalities on each other so that, in equilibrium, the aggregate economy sustains endogenous growth.

Our strategy in describing the model is to focus on the balanced growth path. We then justify this in our stability analysis of Section 3.2, where we establish that in fact there can be no stable transitional dynamics, so that indeed the balanced growth path is the only viable equilibrium.

2.1 Individual behavior

Demography: Individuals are born at time v and the probability that they survive until age $t-v$ is described by the survival function $S(t-v)=e^{-M(t-v)}$ where $M(t-v)=\int_0^{t-v}\mu(\tau)d\tau$ is the cumulative mortality rate and $\mu(t-v)=-S'(t-v)/S(t-v)$ is the hazard rate or instantaneous probability of death. $S'(s)\equiv dS(s)/ds<0$, that is, the probability of survival decreases as the individual ages. Naturally, $S(0)=1$, and $S(D)=0$, where D is the maximum attainable life time, which may be finite or infinite.

Utility and budget constraint: The discounted expected life time utility of an individual born at time v is given by:

$$E\Lambda(v)=\int_v^{v+D}U\left(C(v,t)\right)\cdot e^{-\rho(t-v)-M(t-v)}dt, \tag{2.1}$$

where $C(v,t)$ is consumption at time t of an individual born at time v, ρ is the pure rate of time preferences and $M(t-v)$ is the cumulative mortality rate outlined above. The individual's total discount rate is given by the sum of the pure rate of time preferences and the instantaneous probability of death: $\rho+\mu(t-v)$, which varies with age.[7]

Each individual supplies one unit of labor inelastically, and is assumed to make consumption and savings decisions such that he/she maximizes his/her discounted life time utility, (2.1), subject to the budget constraint:

$$A_t(v,t) \equiv \frac{\partial A(v,t)}{\partial t} = \left(r+\mu(t-v)\right)A(v,t)+w(v,t)-C(v,t), \tag{2.2}$$

where $A(v,t)$ are financial assets and $w(v,t)$ is the wage rate, both at time t, of an individual born at time v and r is the interest rate, which is constant due to the production structure that we employ; see (2.18a). Along the balanced growth path the wage rate of the individual grows at the growth rate of the economy. This allows us to express the wage at any age as $w(v,t)=w(v)e^{\gamma(t-v)}$, where γ is the equilibrium economic growth rate (see below).

Individuals do not have a bequest motive, are not allowed to die indebted, and are born without assets. Hence, assets at birth are zero, i.e. $A(v,v)=0$, and individuals fully annuitize their assets. Annuities are life-insured financial products that pay out, conditional on the survival of the individual. That is, as long as they are alive, individuals receive a premium on the annuities that is equal to the instantaneous probability of death, $\mu(t-v)$.[8] In return, when the individual dies, all remaining assets flow to the annuity firm. The overall rate received on annuities is, therefore, equal to $r+\mu(t-v)$.

Optimal consumption: In addition to the budget constraint (2.2), the agent must satisfy the transversality condition $A(v,v+D)=0$. That is, in the absence of a bequest motive, individuals want to ensure that $A(v,v+D)\geq 0$ and, in light of the mortality risk, the annuity firm wants to ensure that $A(v,v+D)\leq 0$. The only feasible solution in this regard is $A(v,v+D)=0$.

As a final component for the individual decision making process, we follow much of the

[7] The total discount rate increases if and only if $SS''<(S')^2$, which certainly holds if the mortality function is concave.

[8] This result follows from perfect competition between annuity firms. If competition between annuity firms is less-than-perfect there is a load factor, $0\leq\lambda<1$, on the annuity premium and individuals receive only $\lambda\mu(t-v)$ on their annuities. This is studied in Heijdra and Mierau (2012).

contemporary growth literature and assume an isoelastic utility function:

$$U\left(C(v,t)\right)=\frac{C(v,t)^{1-1/\sigma}-1}{1-1/\sigma} \tag{2.3}$$

where σ is the inter-temporal elasticity of substitution, which we shall assume lies in the range (0,1).[9]

Performing the optimization problem outlined above allows us to write the individual consumption Euler equation as:

$$\frac{\partial C(v,t)/\partial t}{C(v,t)}=\sigma\left(r-\rho\right), \tag{2.4}$$

where we immediately see that consumption growth is constant over the life cycle and, more importantly, independent of the individual's survival structure. The latter is a direct consequence of the existence of perfect annuity markets and was first established by Yaari (1965).

By considering a new born agent, we can express his/ her consumption at any age in terms of consumption at birth, $C\left(v,v\right)$, by solving (2.4) forward from time v:

$$C\left(v,t\right)=C\left(v,v\right)e^{\sigma\left(r-\rho\right)\left(t-v\right)}. \tag{2.5}$$

In order to solve for $C\left(v,v\right)$ we integrate the budget constraint (2.2) forward from time v, impose the transversality condition, $A\left(v,v+D\right)=0$, and use (2.5) to obtain:

$$C(v,v)=\frac{H(v,v)}{\Delta(v,v)} \tag{2.6}$$

where:

$$H\left(v,v\right)\equiv\int_{v}^{v+D}w(\tau)e^{-r\left(\tau-v\right)-M\left(\tau-v\right)}d\tau \tag{2.7a}$$

is the discounted value of future labor income (human wealth) of a new born and:

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[9] See e.g. Guvenen (2006) who summarizes much of the literature, which with few exceptions places σ well within the range (0,1).

$$\Delta(v,v) \equiv \int_v^{v+D} e^{-((1-\sigma)r+\sigma\rho)(\tau-v)-M(\tau-v)} d\tau \qquad (2.7b)$$

is the inverse of his/her marginal propensity to consume out of total wealth.[10] The expressions in (2.7) immediately reveal that an increase in the mortality rate leads to a decrease in human capital and an increase in the marginal propensity to consume. Both of these effects can be traced back to the fact that a higher mortality rate implies heavier discounting of the future.

2.2 Aggregate behavior

Aggregate demography: At every instant, a cohort of size $P(v,v) = \beta P(v)$ is born, where $P(v,v)$ is the size of the cohort, $P(v)$ is the size of the total population at time v, and β is the constant birth rate. Given the survival function, the number of individuals of cohort v still alive at time t is equal to $P(v,t) = \beta P(v) e^{-M(t-v)}$. At every instant $\bar{\mu} P(t)$ individuals die, where $\bar{\mu}$ is the average mortality over all cohorts.[11] Abstracting from migration, the growth rate of the population is equal to $\beta - \bar{\mu} = n$.

From the perspective of the generation born at time v, the population at time t is equal to $P(t) = P(v)e^{n(t-v)}$. Alternatively, we can define the total population at time t as the sum of all the surviving cohort members: $P(t) = \beta \int_{t-D}^{t} P(v) e^{-M(t-v)} dv$. Equating these two measures of $P(t)$ yields the demographic steady-state (see Lotka (1998, p. 60)):

$$\frac{1}{\beta} = \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} dv. \qquad (2.8)$$

That is, (2.8) is a constraint that binds the birth rate, mortality structure and the overall population growth rate in such a way that the population is stable.[12]

The relative weight of each cohort is given by:

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[10] Consumption at any other age is given by $C(v,t) = \left(A(v,t) + H(v,t)\right) / \Delta(v,t)$. The term corresponding to $A(v,t)$ is absent from (2.6) because assets at birth are zero [i.e. $A(v,v) = 0$].

[11] Formally we can write the average mortality rate as: $\bar{\mu} \equiv \int_{t-D}^{t} \mu(t-v) P(v,t) / P(t) dv.$

[12] Note that a stable population may still grow. A stationary population, in contrast, is one that is stable and does not grow (Lotka, 1998). This would be a population with $n = 0$ in our case.

$$\frac{P(v,t)}{P(t)} = \beta e^{-n(t-v)-M(t-v)} \equiv p(t-v) \tag{2.9}$$

the dynamics of which are:

$$\frac{p_t(t-v)}{p(t-v)} \equiv \frac{\partial p(t-v)/\partial t}{p(t-v)} = -\left[n+\mu(t-v)\right]. \tag{2.10}$$

Equation (2.10) highlights that the decline in the relative cohort size reflects both its mortality and the overall population growth rate. Being dependent on $(t-v)$, it depends only on age and is independent of calendar time.

While the economic structure of the model depends heavily on the demographic structure, the reverse is not true. That is, neither mortality nor fertility is assumed to depend on the consumption or wealth of the individuals. There is, however, an extensive literature dealing with how a simple demographic structure depends on the economic environment. Manuelli and Seshadri (2009), for instance, use the Barro and Becker (1989) model to study how fertility and mortality are affected by the economic and institutional structure of an economy.[13] Here we focus on an exogenous, but realistic, demographic structure and analyze how different types of demographic change affect the economy-wide growth rate (see section 4.2).

Aggregate quantities: Employing the following generic aggregator function we can obtain the aggregate per capita equivalents of the individual quantities defined above:

$$x(t) \equiv \int_{t-D}^{t} p(t-v)\, X(v,t)\, dv = \beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} X(v,t)\, dv, \tag{2.11}$$

where $x(t)$ is the aggregate per capita value of $X(v,t)$. Taking the time derivative of (2.11) we can express the evolution of $x(t)$ as:

$$\dot{x}(t) = \beta X(t,t) + \int_{t-D}^{t} p(t-v)\, X_t(v,t)\, dv - nx(t) - \int_{t-D}^{t} \mu(t-v)\, p(t-v)\, X(v,t)\, dv \tag{2.12}$$

where we have used the fact that $p(0)=\beta$, $p(D)=0$, as well as (2.10).

[13] Without being exhaustive, other key references in this area include Doepke (2004), Soares (2005), and Cervelatti and Sunde (2005).

Aggregate consumption: Straightforward application of (2.11) implies that aggregate per capita consumption is given by: $c(t) \equiv \int_{t-D}^{t} p(t-v) C(v,t) dv$. Using (2.12) in combination with (2.4), we can write the dynamics of $c(t)$ as:

$$\dot{c}(t) = (\sigma[r-\rho]-n)c(t) + \beta C(t,t) - \int_{t-D}^{t} \mu(t-v)\, p(t-v) C(v,t) dv, \tag{2.13}$$

Using (2.4) once more allows us to express (2.13) more compactly:

$$\frac{\dot{c}(t)}{c(t)} = \frac{\partial C(v,t)/\partial t}{C(v,t)} - \frac{\Phi(t)}{c(t)} \tag{2.14}$$

where:

$$\Phi(t) \equiv \int_{t-D}^{t} \mu(t-v)\, p(t-v) C(v,t) dv - \beta C(t,t) + nc(t) \tag{2.15}$$

is the generational turnover term. It measures the reduction in aggregate per capita consumption due to the arrival of new agents without assets together with the departure of old agents with assets. This brings us to the first proposition:

> **Proposition 1:** *Along the balanced growth path, the generational turnover term is positive as long as the economic growth rate is less than the growth rate of consumption, i.e,* $\gamma < \sigma(r-\rho)$.

Proof: See Appendix A.

Aggregate assets: Using (2.11), we define aggregate per capita assets as: $a(t) \equiv \int_{t-D}^{t} p(t-v) A(v,t) dv$. Applying (2.12), and substituting (2.2) and (2.10) allows us to express the aggregate capital accumulation process as:

$$\dot{a}(t) = (r-n)a(t) + w(t) - c(t). \tag{2.16}$$

The aggregate assets accumulation process in (2.16) differs from the individual asset accumulation process, (2.2), because (a) the premium received on annuities, $\mu(t-v)A(v,t)$, is a transfer from those who die to those who survive, and (b) the growing population is taken into account.

2.3 Firms

Individual firms: There are N identical firms that each produce according to a Cobb-Douglas production function: $Y_i(t) = Z(t)K_i(t)^{\alpha} L_i(t)^{1-\alpha}$, where $Y_i(t)$ is individual output, $K_i(t)$ is individual capital, $L_i(t)$ is individual labor demand, $Z(t)$ is the aggregate level of technology in the economy and α is the capital share of output. In per capita terms the production function can be expressed as: $y_i(t) = Z(t)k_i(t)^{\alpha}$, where $y_i(t)$ and $k_i(t)$ are output and capital per capita, respectively. Assuming that both capital and labor are paid their marginal products, the equilibrium interest rate and wage rate are determined by:

$$r(t) = \alpha Z(t)k_i(t)^{\alpha-1} - \delta, \tag{2.17a}$$

$$w(t) = (1-\alpha)Z(t)k_i(t)^{\alpha}, \tag{2.17b}$$

where δ is the depreciation rate.

Aggregate production: The inter-firm productive externality is given by $Z(t) \equiv Zk(t)^{1-\alpha}$ so that the aggregate per-capita production function is of the AK-type (see, Romer, 1986) $y(t) \equiv Zk(t)$, where Z is the technology index. Taking account of the aggregate production externality, equilibrium factor prices are:

$$r = \alpha Z - \delta, \tag{2.18a}$$

$$w(t) = (1-\alpha)Zk(t). \tag{2.18b}$$

validating our assumption of a constant return to capital and growing wage rate.

3. Equilibrium

We now derive the economy-wide equilibrium and describe its properties. In equilibrium, both the labor and the capital market must clear. For the equilibrium to be viable it must satisfy the optimal decisions made by the households, as well as the transversality condition on individual asset accumulation. We assume that all individuals are employed, so that labor market clearance is implied by equating the total population with the total labor force. Likewise, because productive capital is the only asset in the economy, capital market clearance is implied by setting aggregate assets equal to total capital $A(t)=K(t)$. In per capita terms this becomes $a(t)=k(t)$, implying furthermore that $\dot{a}(t)=\dot{k}(t)$.

3.1 Existence

Using the capital market clearance condition in (2.16) permits us to write the aggregate per capita capital accumulation process as:

$$\dot{k}(t)=Zk(t)-c(t)-(\delta+n)k(t) \tag{3.1}$$

so that (3.1) in combination with (2.14) describes the equilibrium dynamics of the model. Along the equilibrium growth path the economy grows at a rate:

$$\gamma(t)=\frac{\dot{k}(t)}{k(t)}=r-n+(1-\alpha)Z-\frac{c(t)}{k(t)}. \tag{3.2}$$

We can rewrite (3.2) further by noting that $c(t)/k(t)=\left(c(t)/w(t)\right)\left(w(t)/k(t)\right)$ and using the factor price relation in (2.18b):

$$\gamma(t)=r-n+(1-\alpha)Z\left[1-\frac{c(t)}{w(t)}\right]. \tag{3.3}$$

It remains to determine the value of $c(t)/w(t)$. We know that along the equilibrium growth path wages grow at the common growth rate, hence, we can write aggregate per capita consumption as:

$$\frac{c(t)}{w(t)}=\int_{t-D}^{t}p(t-v)\frac{C(v,v)}{w(v)}e^{(\sigma(r-\rho)-\gamma)(t-v)}dv. \tag{3.4}$$

Using (2.6), (2.7) and (2.9) in (3.4) allows us to write the growth rate of the economy implicitly as:

$$\gamma=(r-n)+(1-\alpha)Z\times\left[1-\frac{\varphi(r-\gamma)}{\varphi((1-\sigma)r+\sigma\rho)}\frac{\varphi(\gamma+n-\sigma(r-\rho))}{\varphi(n)}\right]\equiv f(\gamma), \tag{3.5}$$

where $\varphi(x) \equiv \int_0^D e^{-xs-M(s)} ds$ and the demographic steady state (2.8) has been used to eliminate β.[14] Inspection of (3.5) reveals that there exists an equilibrium for which $\gamma = r - n$ and this brings us to the second proposition:

> **Proposition 2:** *The equilibrium for which $\gamma = r - n$ is inconsistent (and therefore infeasible) because it violates the transversality condition on the individual asset accumulation process.*

Proof: see Appendix A.

A similar result, although derived somewhat differently, is obtained by Bruce and Turnovsky (2011).

For what follows we make the assumption that individuals are relatively patient and that the economy is dynamically efficient (i.e. $r > \rho > n$). This allows us to derive the following proposition:

> **Proposition 3:** *There exists a consistent equilibrium growth rate γ^* for which $\gamma^* < \sigma(r - \rho)$ holds.*

Proof: see Appendix A

Intuitively, if $\gamma^* \geq \sigma(r - \rho)$ no individual would ever hold positive assets, which means that it is impossible to sustain the closed economy. As a corollary, note that the result of Proposition 3 implies that, in equilibrium, the generational turnover term is always positive (see Proposition 1). Proposition 3 is related to Corollary 5 of d'Albis and Augeraud-Véron (2009, p.468) in which the same result is established for an overlapping generations model populated with individuals that have a rectangular survival function (i.e. individuals live for certain until time D). d'Albis and Augeraud-Véron also find an infinite number of other growth rates. However, the underlying dynamic process assures that, asymptotically, only the real root (our γ^*) is relevant (see their discussion surrounding Equation (37), p. 470). Hence, our proposition extends their asymptotic

[14] The properties of the $\varphi(x)$ function are discussed in more detail in Appendix A. In (3.5) we also use the fact that, along the equilibrium growth path, only age matters but not the time at which an individual was born. To see this note that, for instance, $C(v,v)/w(v)$ in (2.6 depends only on $\tau - v$ but not τ ; see also Prop. 1 in Mierau and Heijdra, (2012).

result to the case of a convex survival function.

It is interesting to observe that the upper bound of the consistent equilibrium growth rate is equal to the growth rate that would prevail in the representative agent model. This implies that the generational turnover inherent in overlapping generations models causes a negative pull on the economic growth rate. We capture this result in the final proposition:

Proposition 4: *The equilibrium growth rate in the overlapping generations model is less than the equilibrium growth rate in the representative agent model.*

Proof: see Appendix A

3.2 Stability

In general, the description of the dynamics of an overlapping generations model having a realistic demographic structure is very complex. Indeed, in Mierau and Turnovsky (2011) we find that to describe the global dynamics of an overlapping generations model with a neo-classical production structure leads to a 5 dimensional dynamic system consisting of mixed differential-difference equations. In the single-sector endogenous growth setting, d'Albis and Augeraud-Véron (2009) have used a rectangular survival function that allows them to describe the global dynamics as being saddle-point stable by using transcendental functions. However, this procedure becomes intractable for an arbitrary convex survival function. Hence, we adopt a different approach, and apply the second mean value theorem to approximate the generational turnover term. This allows us to summarize the model as a three dimensional dynamic system, the local dynamics of which we can characterize using standard techniques. Here we briefly outline the approach, relegating details to Appendix B.

In general, the macrodynamic equilibrium is described by the pair of equations:

$$\dot{k}(t) = Zk(t) - c(t) - (\delta + n)k(t) \tag{3.7a}$$

$$\dot{c}(t) = \sigma(r - \rho)c(t) - \Phi(t), \tag{3.7b}$$

where (3.7b) is obtained by substituting (2.4) into (2.14), and (3.7a) simply repeats (3.1) for convenience. Recalling (2.15), the generational turnover term is defined as:

$$\Phi(t) \equiv \int_{t-D}^{t} \mu\left(t-v\right) p\left(t-v\right) C\left(v,t\right) dv - \beta C\left(t,t\right) + nc(t). \tag{3.8}$$

To simplify this term, and thereby its underlying dynamics, we apply the second mean value theorem to the first term on the right hand side of (3.8) enabling us to express $\Phi(t)$ as:

$$\Phi(t) = \mu\left(t-v_1\right) \int_{t-D}^{t} p\left(t-v\right) C\left(v,t\right) dv - \beta C\left(t,t\right) + nc(t) \quad v_1 \in (t-D,t), \tag{3.9}$$

Writing

$$\mu\left(t-v_1\right) = \frac{\int_{t-D}^{t} \mu\left(t-v\right) p\left(t-v\right) C\left(v,t\right) dv}{\int_{t-D}^{t} p\left(t-v\right) C\left(v,t\right) dv} \tag{3.10}$$

we see that $\mu(t-v_1)$ is the ratio of the consumption given up by the dying to aggregate consumption and can be interpreted as being the average mortality of consumers over the period $(t-D,t)$. Henceforth, we denote this by $\mu_C(t-v_1)$ to distinguish it from other measures of average μ as computed from (B.9a) and (B.9b) in the Appendix. Using $\mu_C(t-v_1)$ enables us to write (3.8) as:

$$\Phi(t) \equiv (\mu_C(t-v_1)+n)c(t) - \beta C\left(t,t\right). \tag{3.11}$$

In order to describe the dynamics of $C\left(t,t\right) = H\left(t,t\right) / \Delta\left(t,t\right)$ we take the time derivatives of (2.7a) and (2.7b), and apply the mean value theorem again to yield:

$$\dot{H}(t,t) = -w(t) + \left[r + \mu_H(\tau_1 - t)\right] H(t,t) \quad \tau_1 \in (t,t+D) \tag{3.12a}$$

$$\dot{\Delta}(t,t) = -1 + \left[(1-\sigma)r + \sigma\rho + \mu_\Delta(\tau_2 - t)\right] \Delta(t,t) \quad \tau_2 \in (t,t+D) \tag{3.12b}$$

where μ_H and μ_Δ are defined analogously to μ_C and are expressed explicitly in Appendix B.

By defining the stationary variables $x(t) \equiv c(t)/k(t)$ and $y(t) \equiv H(t,t)/k(t)$ we can write the equilibrium dynamic system in (3.7) as:

$$\frac{\dot{x}(t)}{x(t)} = \left[\sigma(r-\rho) - (\mu_C + n)\right] + \frac{\beta}{\Delta(t,t)} \frac{y(t)}{x(t)} - (Z - \delta - n) + x(t) \tag{3.13a}$$

$$\frac{\dot{y}(t)}{y(t)} = -(1-\alpha)Z\frac{1}{y(t)} + r + \mu_H - (Z - \delta - n) + x(t) \tag{3.13b}$$

$$\dot{\Delta}(t,t) = -1 + \left[(1-\sigma)r + \sigma\rho + \mu_\Delta\right]\Delta(t,t). \tag{3.13c}$$

In expressing the dynamics as in (3.13) it is important to note that while the μ terms are functions of time [see e.g. (3.10)], as we show in the Appendix, given the stationarity of the demographic structure and the assumption of the demographic steady state they in fact vary only slightly over time and for practical purposes can be treated as constants. Moreover, being estimates of mortality rates, the μ terms are uniformly small and their equilibrium values can be determined as described in Appendix C.

Linearizing (3.13) around the steady-state, the local dynamics can be expressed as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\Delta} \end{pmatrix} = \begin{pmatrix} \tilde{x} - \frac{\beta}{\tilde{\Delta}}\frac{\tilde{y}}{\tilde{x}} & \frac{\beta}{\tilde{\Delta}} & -\frac{\beta}{\tilde{\Delta}^2}\tilde{y} \\ \tilde{y} & (1-\alpha)\frac{Z}{\tilde{y}} & 0 \\ 0 & 0 & \frac{1}{\tilde{\Delta}} \end{pmatrix} \begin{pmatrix} x - \tilde{x} \\ y - \tilde{y} \\ \Delta - \tilde{\Delta} \end{pmatrix}, \tag{3.14}$$

where tildes indicate the steady-state values of dynamic variables and we have dropped the time indices to avoid cluttering the notation. To establish the stability characteristics of the system (3.14) we must analyze its three eigenvalues, $\lambda_1, \lambda_2, \lambda_3$. From (3.14) we see that these eigenvalues are all positive, and the system therefore locally unstable, if and only if

$$\lambda_1 \equiv \frac{1}{\tilde{\Delta}} > 0 \tag{3.15a}$$

$$\lambda_2 + \lambda_3 \equiv (1-\alpha)\frac{Z}{\tilde{y}} + \tilde{x} - \frac{\beta}{\tilde{\Delta}}\frac{\tilde{y}}{\tilde{x}} > 0 \tag{3.15b}$$

$$\lambda_2\lambda_3 \equiv (1-\alpha)\frac{Z}{\tilde{y}}\left(\tilde{x} - \frac{\beta}{\tilde{\Delta}}\frac{\tilde{y}}{\tilde{x}}\right) - \frac{\beta}{\tilde{\Delta}}\tilde{y} > 0 \tag{3.15c}$$

By definition, $\tilde{\Delta} > 0$ and it is straightforward to show that for a feasible equilibrium growth rate (i.e.

$\gamma^* < \sigma(r-\rho)$), (3.15b) holds as well.[15] The sign of (3.15c) cannot be definitively determined analytically. However, numerical simulations for a very general survival function and a wide variety of underlying parameter values reveal that for any plausible parameter set it too is positive. In this case, all three eigenvalues are positive, indicating that the equilibrium dynamics (3.13) are locally unstable and that, therefore, the only viable equilibrium is for the system always to be on its balanced growth path, as in Romer's (1986) original representative agent version of the model.[16]

Our result contrasts somewhat with the related work of d'Albis and Augeraud-Véron (2009), who find the growth rate of capital to have a transitional path characterized by a saddle-point property. There are key differences in the underlying assumptions that account for the disparity. In addition to assuming a rectangular survival function, d'Albis and Augeraud-Véron assume that the economy starts from a fixed point in time, with the given initial distribution of wealth at that instant being the source of the dynamics of the capital growth rate. In contrast, like Blanchard (1985), the starting point of our economy is in the infinite past and is therefore irrelevant insofar as the equilibrium growth rate is concerned. Moreover, with the arbitrary convex survival function, we are constrained to analyzing the local dynamics around the relevant equilibrium balanced growth path. Like the standard Romer (1986) model and the Blanchard extension, we find that this equilibrium growth path, being disconnected from any given initial point, is locally unstable.

4. Numerical Simulations

Having established the formal properties of the model insofar as possible, to obtain further insights we resort to numerical simulations. In the first exercise, we study how the demographic structure of the model affects the economic growth rate. The second simulation investigates how a changing demographic structure influences the natural rate of wealth inequality. But before reporting the simulations, we outline how the model is parameterized and illustrate some of its basic properties.

[15] In this case since $\lambda_1 > 0, \lambda_2 + \lambda_3 > 0$, we know that at least two of the roots are positive.

[16] In the case of the Blanchard model it is straightforward to establish that all three eigenvalues are positive.

4.1 Model parameterization

To parameterize the model we require an explicit survival function. To this end, we employ the very general function proposed by Boucekkine *et al.* (2002):

$$S(t-v) \equiv e^{-M(t-v)} = \frac{\mu_0 - e^{\mu_1(t-v)}}{\mu_0 - 1}, \qquad \text{(for } 0 \le t-v \le D\text{)}, \qquad \mu_0 > 1,\ \mu_1 > 0, \tag{4.1}$$

where the maximum attainable age, determined by $S(t-v)=0$, is $D = \ln \mu_0 / \mu_1$. In keeping with the terminology of Boucekkine *et al.* we refer to μ_0 as "youth mortality" and μ_1 as "old age mortality". We estimate the two parameters by nonlinear least squares, using US cohort data for 2006.[17] Our estimated results in Table 1 highlight that we obtain a tight fit with highly significant parameter estimates. The resulting survival function is illustrated in Figure 1. As we do not consider childhood or education, we normalize the function so that birth corresponds to age 18. Figure 1 confirms that the survival function tracks the actual survival data very well from 18 until 90. Beyond that, the concavity of the function yields a less satisfactory fit. However, only 1.5% of the US population is older than 85 and almost all these individuals are retired and generally inactive in the economy. In order to satisfy the demographic steady-state, (2.8), we set the birth rate such that the population growth rate equals 1%, as is also observed empirically. This leads to a birth rate of 2.24%, which is somewhat higher than the 1.4% that is observed empirically. However, because we neglect migration, the model's birth rate should be interpreted as including a component reflecting migration.

The remaining structural parameters are standard and are set as follows. The elasticity of capital is $\alpha = 0.35$ and the depreciation rate is $\delta = 0.05$. Furthermore, the aggregate level of technology equals $Z = 0.3286$, which yields a real interest rate of 6.7%. With respect to preferences, we set the intertemporal elasticity of substitution to 0.75, consistent with the upper end of the estimates reported by Guvenen (2006). We take $\rho = 0.035$ to be the rate of time preference at birth, which due to the increasing mortality with age implies a discount rate of 0.0388 for the

[17] Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research, Rostock (Germany). Available at www.mortality.org or www.humanmortality.de.

individual of average age.[18]

Equilibrium growth: In Figure 2 we illustrate the equilibrium growth rate by plotting both the left-hand and right-hand sides of equation (3.5). Point A is the actual equilibrium growth rate and point B is the inconsistent equilibrium discussed in Proposition 2. For comparison we have also added point RA, which is the growth rate that would prevail in the representative agent model. In that case the growth rate is equal to the growth rate of individual consumption, i.e. $\gamma = \sigma(r - \rho)$. As the graph shows, the generational structure induces a negative pull on the growth rate of the economy. This finding is consistent with Proposition 4, which states that the growth rate of the overlapping generations model is smaller than the growth rate of the representative agent model. Row 1 of Table 2 indicates that the value of the equilibrium growth rate for the current parametrization equals 1.03 percent.[19]

Stability: The parameterization of the model allows us to calculate the values of the eigenvalues determined in (3.15). Carrying out these calculations, we find $\lambda_1 = 0.0478$, $\lambda_2 = 0.0415$, $\lambda_3 = 0.2680$, so that all eigenvalues are indeed all positive. Hence, any transitional path is locally unstable, so that to remain viable the economy must always be on its balanced growth path, just like the representative agent model of Romer (1986).

As an aside, the parametrization also permits us to charaterize the magnitude of the μ terms in relation to the stationary variables $\tilde{x}$, $\tilde{y}$ and $\tilde{\Delta}$. The stationary variables are, respectively, equal to 0.2640, 3.6142 and 20.9.[20] The implied values of μ_C, μ_H and μ_Δ are, respectively, 0.0184, 0.0034 and 0.0048, confirming our comment that the μ terms are negligible when compared to the stationary variables.

4.2 Growth and the source of demographic change

The general form of the mortality function we have developed allows us to characterize the relationship between the demographic structure and the economic structure. As we have seen from the various propositions, we know that introducing a realistic demographic structure in an otherwise

[18] In Section 4.3 we establish the robustness of our results to the various parameter assumptions that we make.
[19] This is significantly less than the equilibrium growth rate of 2.4% in the representative agent economy.
[20] These are calculated from (B.14a)-(B.14c) in the Appendix.

unchanged endogenous growth model leads to a significantly lower economic growth rate than in the representative agent model. In this section we pursue this issue further, by analyzing how the economic growth rate of the economy responds to changes in the demographic structure.

We begin by posing the question: How does the economic growth rate respond to a 0.5 percentage point increase in the population growth? Our demographic structure provides two channels through which such a change may occur, namely, either an increase in the fertility rate or a decrease in the mortality rate.[21] In the simulation results presented in Rows 3 and 4 of Table 2, we find that these two different sources of demographic change have dramatically different impacts on the economic growth rate.[22] While a 0.5 percentage point increase in the population growth rate induced by a change in the fertility rate leads to 0.06 percentage point *decrease* in the economic growth rate, the same change induced by a decrease in the mortality rate leads to a 0.4 percentage point *increase* in the economic growth rate.[23]

The driving force behind the two contrasting results is the fact that a decrease in mortality acts as an incentive to save, because individuals can benefit longer from their savings, while an increase in the fertility rate depletes the per capita capital stock and, thereby, aggregate savings. As savings are the source of economic development, an increase or decrease in the incentive to save directly translates into changes in the economic growth rate.

As noted in the introduction, the stark difference relates to the observation made by Kelly and Schmidt (1995), and their contrast between the young and the old as being "resource users" and "resource creators", respectively. As we also noted, the fact that a decline in mortality increases savings and growth agrees with both the theoretical and empirical findings of Bloom *et al.* (2007) and Lorentzen *et al.* (2008), who show that countries that have experienced a decline in mortality have simultaneously experienced an increase in savings, and correspondingly, growth. It also is

[21] The demographic changes running through mortality can either be driven by a change in old age or youth mortality. However, because the two changes give almost the same effect, we focus on the former in our analysis. The numerical results for youth mortality are available on request.

[22] From columns 4 to 6 in Table 2 we observe that model is stable also in the new regimes.

[23] It is interesting to note that the results concerning the relationship between mortality and economic growth also surface in much more elaborate models. Krueger and Ludwig (2007), for instance, find a similar result in a model containing endogenous labor supply and an elaborate pension system. In that model, however, it is not possible to provide the same analysis as Section 3 above.

consistent with the cross-sectional analysis of the driving forces behind economic development, by Sala-i-Martin *et al.* (2004). Using a sample of 88 countries they find that economic growth is positively related to life expectancy, but negatively related to fertility.

The contrasting effects of fertility and mortality driven changes in the population growth rate on the economic growth rate may thus account for the mixed empirical evidence concerning the relationship between the population growth rate and the economic growth. That is, while Kelly and Schmidt (1995) find a negligible relation for the 1960s and 1970s, they obtain a negative relation for the 1980s. Likewise, Sala-i-Martin *et al.* (2004) find that the sign of the impact of the population growth rate on economic growth is ambiguous. These differences and easily be explained in terms of the changing demographic structure over time.

Keeping these ambiguous results in mind, we turn to the second question: Is the relationship between a change in the population growth rate and the economic growth rate monotonic? This question is closely related to the analysis of Boucekkine *et al.* (2002) who show that the relationship between the population growth rate and the economic growth rate is *hump-shaped*, regardless of the source of demographic change. They then go on to argue that this hump-shaped relationship can account for the ambiguous empirical evidence on the relation between the population growth rate and the economic growth.

In Figure 3 we illustrate the relationship between the population growth rate and the economic growth rate. In the left panel we depict the relationship when the source of demographic change is due to a change in the fertility rate, while in the right panel we demonstrate the relationship when mortality is the source of demographic change. Both panels indicate that the relationship between the population growth rate and the economic growth rate is *monotonic*, regardless of the source of demographic change. That is, while the left panel indicates a monotonic negative relation between the population growth rate and the economic growth rate when the source of the change is fertility, the right panel indicates that the relationship is monotonically positive if the source of the change is a change in mortality.

The difference between our results and those of Boucekkine *et al.* (2002) suggests that the existence of a hump-shaped relationship between the population growth rate and the economic growth rate crucially depends on the source of endogenous growth. Whereas in the Boucekkine *et al.* (2002) model growth arises from the accumulation of vintage-dependent human capital, in our analysis growth arises from inter-firm investment externalities. However, by focusing on different combinations of changes in fertility and mortality in determining the population growth rate we can equally well account for the ambiguous empirical relationship. That is, a change in the population growth rate driven by an increase in fertility can account for a negative relationship, a change driven by a decrease in mortality can account for a positive relationship and any change driven by a combination of changes in mortality and fertility can account for all the intermediate results.[24]

4.3 Demographic structure and wealth distribution

The observed increase in wealth inequality has recently attracted a lot of attention both in the United States and elsewhere.[25] In an early contribution, Atkinson (1971) argues that there is an inherent wealth inequality in societies due to the changing savings behavior of agents over their life-cycle. In general, macrodynamic models based on identical representative agents cannot account for this source of wealth inequality. The overlapping generations structure of our model, however, enables us to trace out the development of assets over the individual life-cycle. As can be seen in Figure 4, the asset path is hump-shaped over the life-cycle, in the sense that individuals start out with zero assets, then build up assets for intertemporal consumption smoothing and, toward the end of their lives, deplete their assets so as to assure that assets are zero exactly at the maximum attainable life-time, D. Additionally, in Figure 5 we trace out the share of each cohort at different ages of that cohort. As can be seen, the cohort structure in 2006 was such that a substantial part of the population was made up by individuals between 40 and 50.

The fact that individuals at different stages of their life-cycle possess different levels of wealth and that we know the size of the cohorts to which individuals belong, enables us to calculate standard wealth inequality measures, such as the Gini coefficient. Using this metric, we turn to the

[24] Bruce and Turnovsky (2011) also examine the relationship between population growth and economic growth and obtain monotonic relationships. In their case, where the increased population growth rate is due to lower mortality the sign depends upon the assumptions one makes about the ratio of working time to retirement. While the model is also based on the Romer technology, as noted previously it assumes a different survival function.

[25] For an overview see Atkinson *et al.* (2011) and the references therein.

final question of our analysis: How does wealth inequality change in the wake of a demographic change? Although a large literature exists trying to replicate the observed wealth inequality (for an overview see Cagetti and De Nardi (2008)), relating changes in inequality to the change of the demographic structure is, to the best of our knowledge, new.

Before analyzing the changes in inequality we establish the benchmark inequality in column 9 of Table 2. For our parameterized model we find the Gini coefficient of wealth inequality to be 0.37. At this point we wish to stress that our aim is not to replicate the Gini coefficient of the United States (which is actually 0.80) but rather to analyze how it changes with the demographic structure. Hence, it is the direction of the change that is important rather than its absolute value. Furthermore, our Gini coefficient indicates the degree of inequality inherent in an economy purely due its age composition and abstracting from any within-cohort inequality. In this sense, it is the "natural rate of wealth inequality".[26]

Rows 3 and 4 of Table 2 report the value of the Gini coefficient resulting from 0.5 percentage point increases in the population growth rate driven by fertility and mortality, respectively. As can be seen, a fertility driven change in the population growth rate increases the Gini coefficient only marginally to 0.38, while if it is mortality driven the Gini coefficient increases dramatically to 0.46.

The substantial difference in the increase in inequality following a demographic change can be traced back to the life-cycle pattern of savings. An increase in the birth rate increases the number of young individuals but leaves the life-cycle savings pattern of the individuals unchanged. Hence, by increasing the birth rate, inequality increases only due to the presence of relatively more young individuals who have much fewer assets than do older agents. In contrast, a decrease in the mortality rate changes both the relative distribution of cohort sizes and the life-cycle savings pattern. Due to the longer life-span, individuals will save more for life-cycle purposes, so that the dispersion between asset holdings at different moments of the life-cycle increases substantially. In addition, more agents are alive who are at the top of their life-cycle savings. The combination of these two factors lead to the large observed increase in wealth inequality after a drop in the mortality rate.

[26] The natural rate of wealth inequality is related to the discussion of fair versus unfair inequality (See, for instance, Almås *et al.* (2011) and Devooght (2007)). In contrast to them, however, we do not employ a normative framework but simply notice that a part of the observed (wealth) inequality is due to the age distribution of society.

Since a dominant element of the demographic change that has occurred in the United States over the last half century has been the decline in the adult mortality rate, we conclude that at least part of the observed increase in wealth inequality simply reflects the change in the age profile of the US p opulation.

4.4 Robustness analysis

In addition to the demographic structure, other key parameters include: the productive elasticity of capital α, the depreciation rate δ, the aggregate level of technology Z, the intertemporal substitution elasticity σ and the rate of time preferences ρ. The main numerical outcomes of the paper are: (i) the instability of the transitional dynamics, and the implication that the balanced growth path is the only viable equilibrium, (ii) the stark contrasts between the economic consequences of a birth rate-driven and a mortality-driven change in the population growth rate, (iii) the monotonic relationship between a change in the population growth rate and the economic growth rate, and (iv) the impact of changes in the demographic structure on the natural rate of wealth inequality. These are strong results, and it is therefore important to establish their robustness with respect to alternative assumptions concerning the demographic structure and the economic parameters of the model.[27]

Alternative demography: As an alternative to our base assumption that the demographic structure corresponds to the US cohort data for 2006, we use data for the US cohort that was alive in 1980. For that cohort $\mu_0 = 55.1317$, $\mu_1 = 0.0544$, $\beta = 2.32\%$ and, therefore, $n = 0.96\%$. We replicate the simulation results from Table 2 and Figure 3 in order to assure comparability of the robustness checks with the analysis in the main text. In all cases we find that our previous results remain unchanged. That is, the only viable equilibrium is the balanced growth path, the consequences of demographic change remain precisely as before, and the relationship between the demographic structure and the natural rate of wealth inequality is unchanged. Furthermore, we find that the Gini coefficient in the base equilibrium in 1980 is 0.3549, which is lower than in the

[27] Because of space limitations we do not report the detailed numerical results, all of which are available on request.

baseline model based on 2006 (Gini 0.3650). This lends additional support to our claim that part of the change in wealth inequality can be attributed to changes in the demographic structure.

Economic parameters: Although $\alpha = 0.35$ is generally the consensus value for the productive elasticity of capital in developed countries, a larger value may be more appropriate for less developed countries (Caselli and Feyrer, 2007). Hence, as a second robustness check we use $\alpha = 0.45$ and repeat all the simulations. Furthermore, we have: (i) increased the depreciation rate from $\delta = 0.05$ to $\delta = 0.075$, (ii) reduced the interest rate from 6.7% to 5.5%; (iii) reduced the intertemporal elasticity of substitution from 0.75 to 0.5, and (iv) increased the raw rate of time preference from 0.035 to 0.045. We have also considered intermediate values. In all cases we find that the qualitative results of our base numerical simulations continue to hold.

Most importantly, extensive robustness analysis over all the parameters indicates that the three eigenvalues are always positive, so that the economy always jumps to its balanced growth path equilibrium. Equally importantly, the qualitative relationships between growth and the source of population growth illustrated in Fig. 3 remain virtually unchanged. In short, there is no doubt that our findings are robust across a wide range of variations in parameters.

5. Conclusion

In this paper we have extended the Romer (1986) endogenous growth model to allow for a general demographic structure. In the theoretical part of the paper we have shown that, *inter alia*, the demographic structure inflicts a negative drag on aggregate consumption and that the growth rate that prevails in the demographic model is lower than the growth rate that prevails in the representative agent model. Furthermore, using a novel approach to linearize the model around the balanced growth path, we have established that the model is locally unstable and that, therefore, the economy is permanently on its equilibrium path.

In the first numerical analysis, we have established that the consequences of a change in the population growth rate for the economic growth rate differ substantially depending on the source of the demographic change. That is, while an increase in the population growth rate driven by an increase in the fertility rate has a negative impact on economic growth, a change driven by a drop in the mortality rate has a positive impact. We use this difference to reconcile the ambiguous empirical evidence on the relationship between the population growth rate and the economic growth rate.

In the second numerical analysis, we have studied the relationship between a changing demographic structure and the "natural rate of wealth inequality". In this regard we have found that an increase in the fertility rate leads to a slight increase in inequality while a decrease in the mortality rate leads to a substantial increase in inequality. The results imply that at least part of the dramatic increase in wealth inequality of the last decades can attributed to demographic factors.

While our model is stylized, it can be extended in various directions. One obvious extension is to allow for labor supply on both the intensive and extensive margins, to study how the changing demographic environment affects life-cycle decisions concerning retirement decisions. Second, using the approach we have employed for linearizing the model it would be interesting to consider the (local) stability properties of an economy having a neoclassical production structure and yielding transitional dynamics. Finally, by introducing heterogeneity between agents of the same age, the analysis could be extended to shed light on intra-cohort inequality in conjunction with life-cycle behavior.[28]

[28] A natural framework within which to integrate intra-cohort and inter-cohort heterogeneity is the canonical model developed by García-Peñalosa and Turnovsky (2006), which analyzes the growth-income inequality tradeoff in the context of the Romer production technology, but abstracts from the demographic aspects emphasized here.

Table 1

Estimated Survival Function

$S(u)=e^{-M(u)}=\dfrac{\mu_0-e^{\mu_1 u}}{\mu_0-1}+\varepsilon$ where* $\varepsilon \sim i.i.d.(0,\sigma^2)$	
US Cohort	2006
μ_0 (st. dev.)	78.3618 (6.0193)
μ_1 (st. dev.)	0.0566 (0.0011)
Adj. R^2	0.9961

* $I(u \le D)$ is an indicator function that is 1 for $u \le D$ and 0 otherwise.

Table 2

Numerical Simulations

		Demography		Economic Variables				
		L_{18}	n	γ	λ_1	λ_2	λ_3	*Gini*
Baseline Model		78.38	1.00%	1.03%	0.0478	0.0415	0.2680	0.3650
Demographic Shocks								
Increase in fertility rate	$\beta \to 2.57\%$	78.38	1.50%	0.97%	0.0478	0.0390	0.2642	0.3830
Decrease in old age mortality	$\mu_1 \to 0.0443$	95.15	1.50%	1.42%	0.0458	0.0351	0.2591	0.4576

Note that because of the ongoing economic growth we cannot define the capital stock, wage, consumption etc. in the steady-state.

Figure 1
Demography

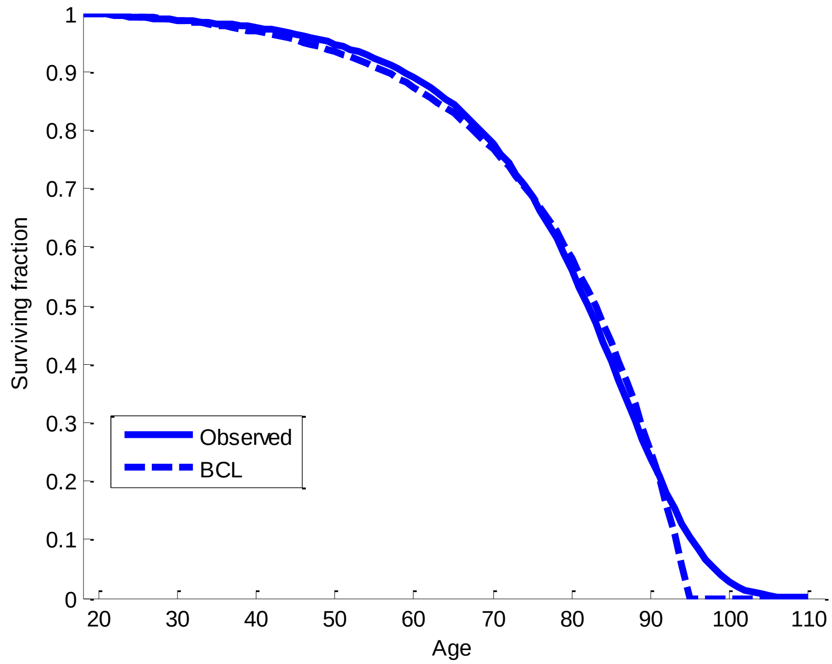


Figure 2
Equilibrium

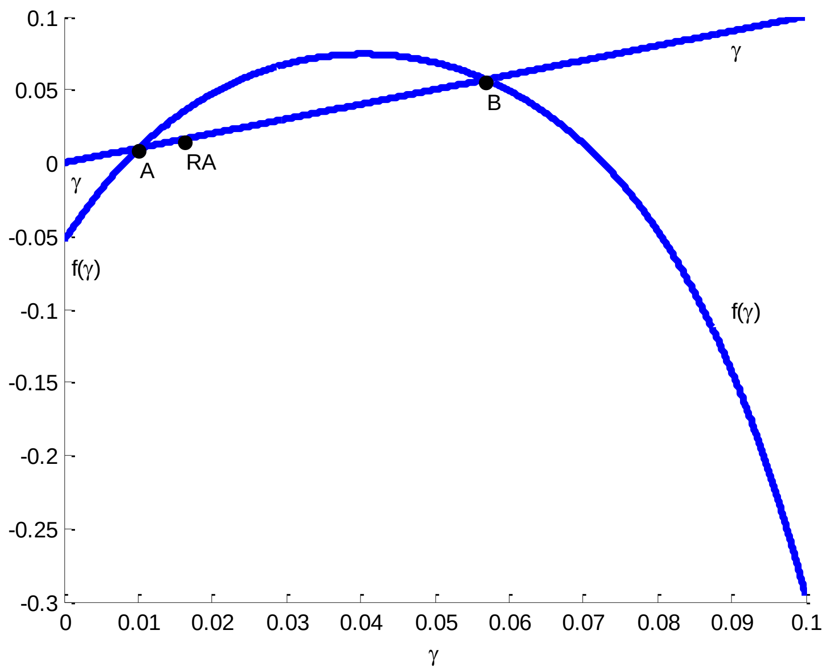
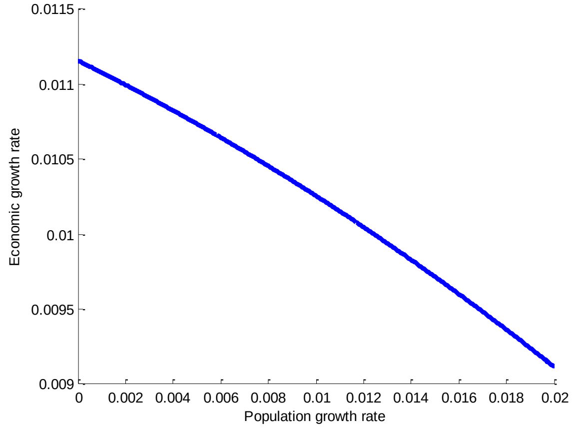
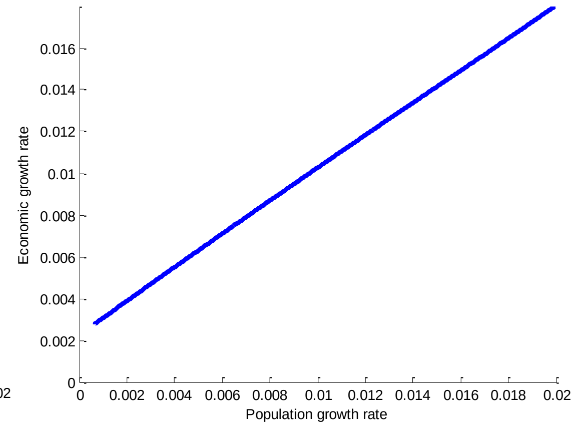


Figure 3

Demographic change and economic growth



3.A Fertility



3.B Mortality

Figure 4

Individual asset profile

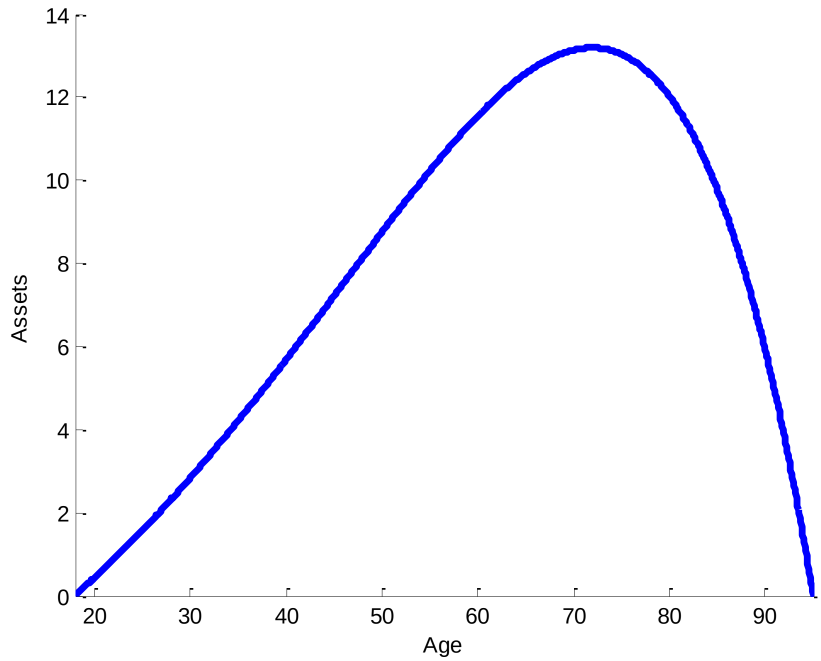
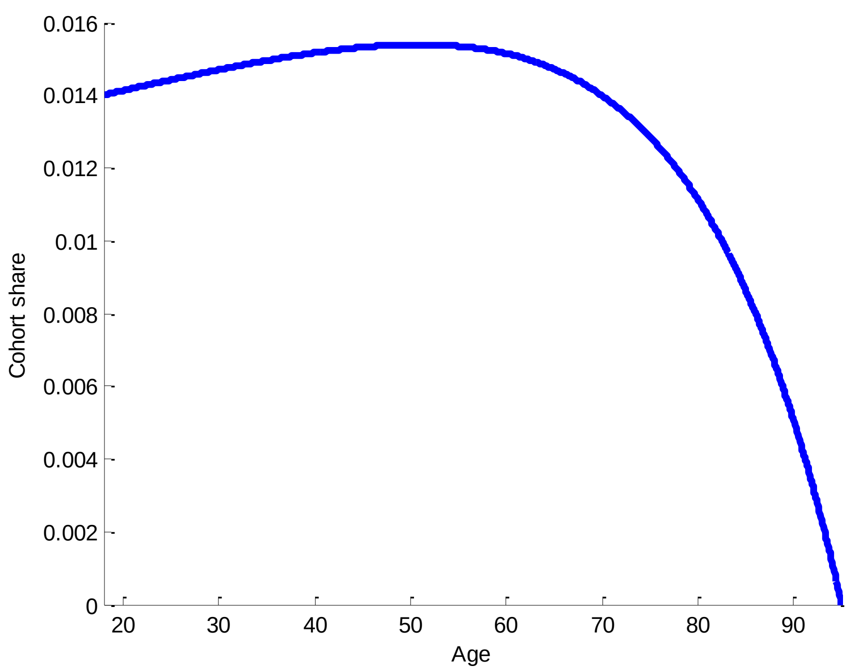


Figure 5

Relative cohort shares



Appendix

A: Proofs of Propositions

Proposition 1: *Along the balanced growth path the generational turnover term is positive as long as the economic growth rate is smaller than the growth rate of consumption, i.e.* $\gamma < \sigma(r-\rho)$.

Proof: We begin by noting that the generational turnover term (2.15) can be rewritten as:

$$\Phi(t) = -\beta \int_{t-D}^{t} S'(t-v) e^{-n(t-v)} C(v,t)\, dv - \beta C(t,t) + nc(t) \tag{P1.1}$$

where we have used the fact that $\mu(t-v) = -S'(t-v)/S(t-v)$ and (2.9). Integrating (P1.1) by parts and simplifying yields:

$$\Phi(t) = -\beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} \left[nC(v,t) + C_v(v,t) \right] dv + nc(t) = -\beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} C_v(v,t)\, dv, \tag{P1.2}$$

where $C_v(v,t)$ is the change in consumption across cohorts at a given point in time. At any moment in time we can write the rate of change of individual consumption as:[29]

$$\dot{C}(v,t) = C_v(v,t) + C_t(v,t). \tag{P1.3}$$

Using (2.4) and noticing that, along the balanced growth path, consumption has to grow at the common growth rate (i.e. $\dot{C}(v,t)/C(v,t) = \gamma$) we can write (P1.3) as:

$$C_v(v,t) = \left(\gamma - \sigma(r-\rho)\right) C(v,t). \tag{P1.4}$$

Using (P1.4) we can rewrite (P1.2) as:

$$\Phi(t) = -\beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} \left(\gamma - \sigma(r-\rho)\right) C(v,t)\, dv = -\left(\gamma - \sigma(r-\rho)\right) c(t), \tag{P1.5}$$

which immediately reveals that as long as $\gamma < \sigma(r-\rho)$ $\Phi(t) > 0$. This completes the proof.

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[29] The rate of change of consumption of $t-v$ year old agents over time is $\lim_{h\to 0} \frac{C(v+h,t+h) - C(v,t)}{h} = C_v + C_t$

Proposition 2: *The equilibrium for which* $\gamma = r - n$ *is inconsistent (and therefore infeasible) because it violates the transversality condition on the individual asset accumulation process.*

Proof: We can write aggregate consumption as:

$$c(t) \equiv \int_{t-D}^{t} p(t-v) C(v,t) dv = \int_{t-D}^{t} p(t-v)[(r+\mu(t-v)) A(v,t) + w(t) - A_t(v,t)] dv$$

$$= w(t) + \int_{t-D}^{t} p(t-v)[(r+\mu(t-v)) A(v,t) - A_t(v,t)] dv \qquad \text{(P2.1)}$$

If $\gamma = r - n$ then (3.3) implies that $c(t) = w(t)$, this allows us to write (P2.1) as:

$$\int_{t-D}^{t} [(r+\mu(t-v)) A(v,t) - A_t(v,t)] e^{-n(t-v)-M(t-v)} dv = 0. \qquad \text{(P2.2)}$$

Integrating an individual agent's budget constraint over his lifetime, recognizing that his initial financial wealth is zero, and recalling the transversality condition, yields the his intertemporal budget constraint:

$$\int_{v}^{v+D} \left[w(\tau) - C(v,\tau) \right] e^{-r(\tau-v)-M(\tau-v)} d\tau = 0. \qquad \text{(P2.3)}$$

Substituting the budget constraint from (2.2) into (P2.3) gives:

$$\int_{v}^{v+D} \left[(r+\mu(t-v)) A(v,t) - A_t(v,t) \right] e^{-r(\tau-v)-M(\tau-v)} d\tau = 0. \qquad \text{(P2.4)}$$

Clearly, (P2.2) and (P2.4) can only hold simultaneously if $r = n$ (and $\gamma = 0$). As both r and n are set exogenously, there is no reason for this to be the case. This completes the proof

Proposition 3: *There exists a consistent equilibrium growth rate* γ^* *for which* $\gamma^* < \sigma(r-\rho)$ *holds.*

Proof: We proceed in two steps. We first establish the existence of an equilibrium, and in the second step we show that one of its characteristics is that it is smaller than $\sigma(r-\rho)$.

For the first step we begin by noting that (3.5) can be rewritten as:

$$\Psi(\gamma) \equiv \frac{\gamma + n - r}{(1-\alpha)Z} \varphi\big((1-\sigma)r + \sigma\rho\big)\varphi(n) = \Gamma(\gamma) \tag{P3.1}$$

where:

$$\Gamma(\gamma) \equiv \varphi\big((1-\sigma)r + \sigma\rho\big)\varphi(n) - \varphi(r-\gamma)\varphi\big(n - \sigma(r-\rho) + \gamma\big). \tag{P3.2}$$

and $\varphi(x) \equiv \int_0^D e^{-xs-M(s)} ds$. As $\Psi(\gamma)$ is an increasing, linear, function of γ we know that for there to exist at least one fixed point $\Gamma(\gamma)$ needs to be concave. Straightforward inspection of (P3.1) and (P3.2) reveals that $\gamma = r - n$ is an equilibrium but from Proposition 3 we recall that it is inconsistent. The aim is thus to establish that $\Gamma(\gamma)$ is concave and that $\gamma = r - n$ is not the unique point of intersection.

In order to establish the properties of the $\Gamma(\gamma)$ function we first study the properties of the sub-function $\varphi(x)$. Specifically, we see

$$\begin{aligned} &a.)\, \varphi'(x) = -\int_0^D s e^{-xs-M(s)} ds < 0 \quad && b.)\, \varphi''(x) = \int_0^D s^2 e^{-xs-M(s)} ds > 0 \\ &c.)\, \varphi''(x) > \frac{\varphi'(x)^2}{\varphi(x)} > 0 && d.)\, \frac{\varphi''(x)\varphi(x) - \varphi'(x)^2}{\varphi(x)^2} > 0 \end{aligned} \tag{P3.3}$$

where properties a.) and b.) follow from straightforward differentiation, while property c.) is a consequence of the Cauchy-Schwarz inequality. To see this, write the inequality in the form:

$$\int_0^D f^2(s)ds \int_0^D g^2(s)ds \geq \left[\int_0^D f(s)g(s)ds\right]^2$$

and define the characteristic functions as $f(s) = s e^{-[xs-M(s)]^{1/2}}$ and $g(s) = e^{-[xs-M(s)]^{1/2}}$. Property d.) follows immediately from c.).

Using these properties of the $\varphi(x)$ function we can determine the properties of $\Gamma(\gamma)$

1.) For $\gamma = r - n$, $\Psi(\gamma) = 0$ while, similarly we can establish that:

$$\Gamma(r-n) = \varphi\big((1-\sigma)r + \sigma\rho\big)\varphi(n) - \varphi\big((1-\sigma)r + \sigma\rho\big)\varphi(n) = 0 \tag{P3.4}$$

Hence, the equilibrium condition in (P3.1) is satisfied. However, as shown in Proposition 2 $\gamma = r - n$ violates the transversality condition and can, therefore, be ignored.

2.) For $\gamma = \sigma\left(r-\rho\right)$ we can establish that:

$$\Gamma(\sigma\left(r-\rho\right)) = \varphi\left(\left(1-\sigma\right)r+\sigma\rho\right)\varphi(n) - \varphi\left(\left(1-\sigma\right)r+\sigma\rho\right)\varphi\left(n\right) = 0 \qquad \text{(P3.5)}$$

However, $\Psi(\gamma) \neq 0$ so that $\gamma = \sigma\left(r-\rho\right)$ does not constitute an equilibrium.

To proceed further, we study the curvature of the $\Gamma(\gamma)$ function. Its first derivative is:

$$\Gamma'(\gamma) = \varphi'(r-\gamma)\varphi\left(n-\sigma(r-\rho)+\gamma\right) - \varphi(r-\gamma)\varphi'\left(n-\sigma(r-\rho)+\gamma\right) \qquad \text{(P3.6)}$$

Evaluating this at the roots of the $\Gamma(\gamma)$ function identified in (P3.4) and (P3.5) yields:

$$\Gamma'(r-n) = \varphi'(n)\varphi\left((1-\sigma)r+\sigma\rho\right) - \varphi(n)\varphi'\left((1-\sigma)r+\sigma\rho\right) \qquad \text{(P3.7)}$$

$$\Gamma'(\sigma(r-\rho)) = \varphi(n)\varphi'\left((1-\sigma)r+\sigma\rho\right) - \varphi'(n)\varphi\left((1-\sigma)r+\sigma\rho\right) \qquad \text{(P3.8)}$$

and it follows that $\Gamma'(\gamma)$ switches sign between its two roots (i.e. $\text{sgn}\left[\Gamma'\left(\sigma(r-\rho)\right)\right] = -\text{sgn}\left[\Gamma'\left(r-n\right)\right]$). To determine $\text{sgn}\left[\Gamma'\left(r-n\right)\right]$ note that from property d.) of (P3.3) we know that $\varphi'\left(x'\right)/\varphi\left(x'\right) > \varphi'\left(x\right)/\varphi\left(x\right)$ where $x' > x$ and hence:[30]

$$\frac{\varphi'\left((1-\sigma)r+\sigma\rho\right)}{\varphi\left((1-\sigma)r+\sigma\rho\right)} > \frac{\varphi'(n)}{\varphi(n)} \qquad \text{(P3.9)}$$

which implies $\Gamma'\left(\sigma(r-\rho)\right) > 0$ and $\Gamma'\left(r-n\right) < 0$.

The second derivative of $\Gamma\left(\gamma\right)$ is given by:

$$\begin{aligned}\Gamma''(\gamma) = &-\varphi''(r-\gamma)\varphi[n-\sigma(r-\rho)+\gamma] + 2\varphi'(r-\gamma)\varphi'[n-\sigma(r-\rho)+\gamma] \\ &-\varphi(r-\gamma)\varphi''[n-\sigma(r-\rho)+\gamma]\end{aligned} \qquad \text{(P3.10)}$$

Using property c.) of (P3.3) we can establish:

$$\begin{aligned}\Gamma''(\gamma) < &-\frac{\varphi'(r-\gamma)^2}{\varphi(r-\gamma)}\varphi\left(n-\sigma(r-\rho)+\gamma\right) + 2\varphi'(r-\gamma)\varphi'\left(n-\sigma(r-\rho)+\gamma\right) \\ &-\varphi(r-\gamma)\frac{\varphi'\left(n-\sigma(r-\rho)+\gamma\right)^2}{\varphi\left(n-\sigma(r-\rho)+\gamma\right)}\end{aligned}$$

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[30] To see this take the derivative of $\varphi'\left(x\right)/\varphi\left(x\right)$ and apply property d.).

and, therefore:

$$\Gamma''(\gamma) < -\frac{1}{\varphi(r-\gamma)\varphi\left(n-\sigma(r-\rho)+\gamma\right)}\left[\varphi'(r-\gamma)\varphi\left(n-\sigma(r-\rho)+\gamma\right)-\varphi(r-\gamma)\varphi'\left(n-\sigma(r-\rho)+\gamma\right)\right]^2 < 0$$

which can be expressed more compactly as:

$$\Gamma''(\gamma) < -\frac{\Gamma'(\gamma)^2}{\varphi(r-\gamma)\varphi\left(n-\sigma(r-\rho)+\gamma\right)} < 0. \qquad \text{(P3.11)}$$

and implies that $\Gamma\left(\gamma\right)$ is concave. Thus, there exists an equilibrium growth rate:

$$\Psi(\gamma^*) = \Gamma(\gamma^*)\,. \qquad \text{(P3.12)}$$

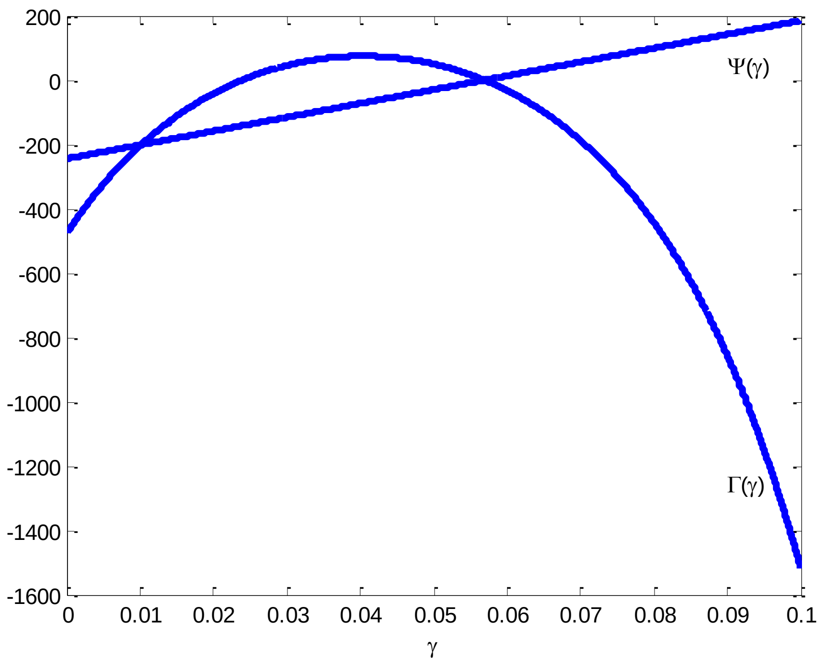
This completes the first part of the proof.

To establish the second part of the proposition (that $\gamma^* < \sigma\left(r-\rho\right)$) we simply note that $\Psi\left(\sigma\left(r-\rho\right)\right) < 0$ while $\Gamma\left(\sigma\left(r-\rho\right)\right) = 0$. This completes the proof.

In order to clarify the above arguments, we graph the $\Psi\left(\gamma\right)$ and $\Gamma\left(\gamma\right)$ functions in Fig. A.1.

Figure A.1

Equilibrium



Proposition 4: *The growth rate that prevails in the overlapping generations model is less than the growth rate that prevails in the representative agent model.*

Proof: Observe that the growth rate in the representative agent model equals the growth rate of individual consumption. Using (2.4) we establish that the growth rate in the representative agent case is:

$$\gamma^{RA} \equiv \sigma\left(r-\rho\right), \qquad \text{(P4.1)}$$

where RA designates Representative Agent. From Proposition 3 we know that for the consistent growth rate it holds that $\gamma^{*} < \sigma\left(r-\rho\right)$, hence, $\gamma^{*} < \gamma^{RA}$. This completes the proof.

B: Stability

The dynamics of the macroeconomic equilibrium can be summarized in the following form:

$$\dot{k}(t) = Zk(t) - c(t) - (\delta + n)k(t) \qquad \text{(B.1a)}$$

$$\dot{c}(t) = \sigma(r-\rho)c(t) - \Phi(t) \qquad \text{(B.1b)}$$

where $r \equiv \alpha Z - \delta$ and $w(t) \equiv (1-\alpha)Zk(t)$ and:

$$\Phi(t) \equiv \int_{t-D}^{t} \mu\left(t-v\right) p\left(t-v\right) C\left(v,t\right) dv - \beta C\left(t,t\right) + nc(t). \qquad \text{(B.2)}$$

Using the second mean value theorem,[31] we may write (B.2) as:

$$\Phi(t) \equiv \mu_C\left(t-v_1\right) \int_{t-D}^{t} p\left(t-v\right) C\left(v,t\right) dv - \beta C\left(t,t\right) + nc(t) \;\; v_1 \in (t-D,t), \qquad \text{(B.3)}$$

where:

$$\mu_C\left(t-v_1\right) = \frac{\int_{t-D}^{t} \mu\left(t-v\right) p\left(t-v\right) C\left(v,t\right) dv}{\int_{t-D}^{t} p\left(t-v\right) C\left(v,t\right) dv} \qquad \text{(B.4)}$$

is the ratio of the consumption given up by the dying to aggregate per-capita consumption. We show

[31] For any real valued function $f(x)$ on the interval $[a,b]$ and function $g(x)$ that is integrable and does not change sign over the interval (a,b) there exists a value $c \in (a,b)$ such that $\int_a^b f\left(x\right) g\left(x\right) dx = f\left(c\right) \int_a^b g\left(x\right) dx$.

below that $\mu_C\left(t-v_1\right)$ varies only very slightly over time, enabling us to treat it as essentially constant. Equally importantly, being a weighted average of mortality rates across cohorts, μ_C is small. Recalling the definition of $c(t)$, we can express (B.3) in the more compact form:

$$\Phi(t) \equiv (\mu_C + n)c(t) - \beta C\left(t,t\right). \tag{B.5}$$

In order to describe the dynamics of $C(t,t)$ we can use the fact that from (2.6) we know that:

$$C(t,t) \equiv \frac{H(t,t)}{\Delta(t,t)} \tag{B.6}$$

where we take from (2.7) that:

$$H(t) \equiv H\left(t,t\right) \equiv \int_t^{t+D} w(\tau)e^{-r(\tau-t)-M(\tau-t)}d\tau \tag{B.7a}$$

and:

$$\Delta(t) \equiv \Delta\left(t,t\right) \equiv \int_t^{t+D} e^{(\sigma-1)r(\tau-t)-\sigma\rho(\tau-t)-M(\tau-t)}d\tau, \tag{B.7b}$$

are, respectively, human wealth and the marginal propensity to consume at birth. The dynamics of (B.7a) and (B.7b) are given by:

$$\dot{H}(t) = -w(t) + \left[r + \mu_H(\tau_1 - t)\right]H(t) \quad \tau_1 \in (t, t+D) \tag{B.8a}$$

and:

$$\dot{\Delta}(t) = -1 + \left[(1-\sigma)r + \sigma\rho + \mu_\Delta(\tau_2 - t)\right]\Delta(t) \quad \tau_2 \in (t, t+D) \tag{B.8b}$$

where μ_H and μ_Δ are defined analogously to μ_C:

$$\mu_H\left(\tau_1 - t\right) = \frac{\int_t^{t+D} \mu(\tau-t)w\left(\tau\right)e^{-r(\tau-t)-M(\tau-t)}d\tau}{\int_t^{t+D} w(\tau)e^{-r(\tau-t)-M(\tau-t)}d\tau} \tag{B.9a}$$

$$\mu_\Delta\left(\tau_2 - t\right) = \frac{\int_t^{t+D} \mu(\tau-t)e^{(\sigma-1)r(\tau-t)-\sigma\rho(\tau-t)-M(\tau-t)}d\tau}{\int_t^{t+D} e^{(\sigma-1)r(\tau-t)-\sigma\rho(\tau-t)-M(\tau-t)}d\tau} \tag{B.9b}$$

Using (B.8) and (B.5) we can write the dynamic system in (B.1) as:

$$\dot{k}(t) = Zk(t) - c(t) - (\delta + n)k(t) \tag{B.10a}$$

$$\frac{\dot{c}(t)}{c(t)} = \left[\sigma(r-\rho) - (\mu_C + n)\right] + \beta \frac{H(t)}{\Delta(t)} \frac{1}{c(t)} \tag{B.10b}$$

$$\frac{\dot{H}(t)}{H(t)} = -(1-\alpha)Z \frac{k(t)}{H(t)} + r + \mu_H \tag{B.10c}$$

$$\dot{\Delta}(t) = -1 + \left[(1-\sigma)r + \sigma\rho + \mu_\Delta\right]\Delta(t) \tag{B.10d}$$

From here we can redefine the system in terms of the stationary variables: $x \equiv c/k,\ y \equiv H/k,\ \Delta$ of which the dynamics are:

$$\frac{\dot{x}}{x} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k};\ \frac{\dot{y}}{y} = \frac{\dot{H}}{H} - \frac{\dot{k}}{k} \tag{B.11}$$

and (B.10d) so that system (B.10) can be written as:

$$\frac{\dot{x}(t)}{x(t)} = \left[\sigma(r-\rho) - (\mu_C + n)\right] + \frac{\beta}{\Delta(t)} \frac{y(t)}{x(t)} - (Z - \delta - n) + x(t) \tag{B.12a}$$

$$\frac{\dot{y}(t)}{y(t)} = -(1-\alpha)Z \frac{1}{y(t)} + r + \mu_H - (Z - \delta - n) + x(t) \tag{B.12b}$$

$$\dot{\Delta}(t) = -1 + \left[(1-\sigma)r + \sigma\rho + \mu_\Delta\right]\Delta(t) \tag{B.12c}$$

These three equations form the basis for the local dynamics of the equilibrium.

To show that the μ_i $(i = C, H, \Delta)$ terms are virtually constant over time, we proceed as follows, focusing on μ_C, although the other two cases are analogous. Letting $t - v = s$, (B.4) may be written as

$$\mu_C\left(t - v_1\right) = \frac{\int_0^D \mu(s)\, p(s) C(t-s,t)\, ds}{\int_0^D p(s) C(t-s,t)\, ds} \tag{B.13}$$

Recalling (2.5), we have $C(t-s,t) = C(t-s,t-s)e^{\sigma(r-\rho)s}$. In addition, suppose that consumption were to grow at the time-varying rate $\gamma_C(u)$ over the period $(t-s,t)$. Then

$C(t-s,t-s) = C(t,t)e^{-\int_{t-s}^{t}\gamma_C(u)du}$ and (B.13) can be written as

$$\mu_C\left(t-v_1\right) = \frac{\int_0^D \mu\left(s\right) p\left(s\right) e^{\sigma(r-\rho)s} e^{-\int_{t-s}^{t}\gamma_C(u)du} ds}{\int_0^D p\left(s\right) e^{\sigma(r-\rho)s} e^{-\int_{t-s}^{t}\gamma_C(u)du} ds} \qquad \text{(B.13')}$$

To show that $\left|d\mu_C/dt\right|$ is very small we take the time derivative of (B.13') to obtain

$$\frac{d\mu_C/dt}{\mu_C} = \frac{\int_0^D p\left(s\right) e^{\sigma(r-\rho)s} e^{-\int_{t-s}^{t}\gamma_C(u)du} [\gamma_C(t)-\gamma_C(t-s)]ds}{\int_0^D p\left(s\right) e^{\sigma(r-\rho)s} e^{-\int_{t-s}^{t}\gamma_C(u)du} ds} - \frac{\int_0^D \mu\left(s\right) p\left(s\right) e^{\sigma(r-\rho)s} e^{-\int_{t-s}^{t}\gamma_C(u)du} [\gamma_C(t)-\gamma_C(t-s)]ds}{\int_0^D \mu(s) p\left(s\right) e^{\sigma(r-\rho)s} e^{-\int_{t-s}^{t}\gamma_C(u)du} ds}$$

which can be written more compactly as

$$\frac{d\mu_C/dt}{\mu_C} = \frac{\int_0^D F(s,t)[\gamma_C(t)-\gamma_C(t-s)]ds}{\int_0^D F(s,t)ds} - \frac{\int_0^D \mu\left(s\right) F(s,t)[\gamma_C(t)-\gamma_C(t-s)]ds}{\int_0^D \mu(s)F(s,t)ds} \qquad \text{(B.14)}$$

where $F(s,t) \equiv p\left(s\right) e^{\sigma(r-\rho)s} e^{-\int_{t-s}^{t}\gamma_C(u)du} > 0$. Equation (B.14) simplifies to

$$\frac{d\mu_C/dt}{\mu_C} = \frac{\int_0^D \mu\left(s\right) F(s,t)\gamma_C(t-s)ds}{\int_0^D \mu(s)F(s,t)ds} - \frac{\int_0^D F(s,t)\gamma_C(t-s)]ds}{\int_0^D F(s,t)ds} \qquad \text{(B.14')}$$

Written in this way, we see that the percentage rate of change of μ_C is the difference of two weighted averages of the consumption growth rate, $\gamma_C(t-s)$. If either the consumption growth rate is constant over time or the death rate is uniform across age, (B.14') is zero and μ_C is indeed constant over time.[32] Otherwise, if we let $\gamma_C^{\max}, \gamma_C^{\min}$ denote the maximum and minimum consumption growth rates, (B.14) implies:

[32] Note that the uniformity of the death rate across agents is the core feature of the mortality structure adopted by Blanchard (1985).

$$\left|\frac{d\mu_C/dt}{\mu_C}\right| < \left|\gamma_C^{\max} - \gamma_C^{\min}\right| \tag{B.15}$$

Assuming that the potential variation in the consumption growth rate is restricted to a few percentage points, (B.15) suggests that as a practical matter the percentage variation of μ_C over time is extremely small, and given that μ_C itself is small, μ_C can essentially be treated as constant and independent of t. Analogous arguments apply to μ_H, μ_Δ, thereby enabling us to approximate them all as constants in addressing the dynamic adjustment of the aggregate economy.

Linearizing (B.12a) – (B.12c) around the steady state, the local dynamics can be expressed as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\Delta} \end{pmatrix} = \begin{pmatrix} \tilde{x} - \dfrac{\beta}{\tilde{\Delta}}\dfrac{\tilde{y}}{\tilde{x}} & \dfrac{\beta}{\tilde{\Delta}} & -\dfrac{\beta}{\tilde{\Delta}^2}\tilde{y} \\ \tilde{y} & (1-\alpha)\dfrac{Z}{\tilde{y}} & 0 \\ 0 & 0 & \dfrac{1}{\tilde{\Delta}} \end{pmatrix} \begin{pmatrix} x - \tilde{x} \\ y - \tilde{y} \\ \Delta - \tilde{\Delta} \end{pmatrix} \tag{B.16}$$

where tildes denote the steady-state values of dynamic variables. The three eigenvalues will be positive, and the system therefore unstable, if and only if:

(i) $\quad \tilde{\Delta} > 0,$

(ii) $\quad (1-\alpha)\dfrac{Z}{\tilde{y}} + \tilde{x} - \dfrac{\beta}{\tilde{\Delta}}\dfrac{\tilde{y}}{\tilde{x}} > 0,$

(iii) $\quad (1-\alpha)\dfrac{Z}{\tilde{y}}\left(\tilde{x} - \dfrac{\beta}{\tilde{\Delta}}\dfrac{\tilde{y}}{\tilde{x}}\right) - \dfrac{\beta}{\tilde{\Delta}}\tilde{y} > 0.$

In order to study the dynamics further it is convenient to write the steady-state values of $\tilde{\Delta}, \tilde{x}, \tilde{y}$ in terms of the $\varphi(.)$ functions discussed in Appendix A above. To do this we substitute the balanced growth conditions into (2.7a), (2.7b), (3.2) and (3.5) resulting in

$$\tilde{\Delta} = \varphi\big((1-\sigma)r + \sigma\rho\big) \tag{B.17a}$$

$$\tilde{x} = (1-\alpha)Z\,\frac{\varphi(r-\gamma^*)}{\varphi\big((1-\sigma)r+\sigma\rho\big)}\,\frac{\varphi\big(n-\sigma(r-\rho)+\gamma^*\big)}{\varphi(n)} \tag{B.17b}$$

$$\tilde{y} = (1-\alpha) Z \varphi\left(r - \gamma^*\right), \tag{B.17c}$$

where we have used the demographic steady state $\beta = 1/\varphi(n)$. Straightforward inspection of (B.14a) reveals that condition (i) is always met regardless of the values of the underlying parameters. Using (B.14) we can rewrite condition (ii) above as:

$$\frac{1}{A} + E\frac{A}{B}\frac{C}{D} - \frac{1}{C} > 0 \tag{B.18}$$

where:

$$\begin{aligned} &A = \varphi\left(r-\gamma^*\right) && B = \varphi\left(\left(1-\sigma\right)r + \sigma\rho\right) \\ &D = \varphi\left(n\right) && C = \varphi\left(n - \sigma\left(r-\rho\right) + \gamma^*\right) \\ &E = \left(1-\alpha\right)Z \end{aligned}$$

In order to show that (B.15) holds we can use the fact that if $C > A$ the condition holds for sure. $C > A$ implies that $\varphi\left(n-\sigma\left(r-\rho\right)+\gamma\right) > \varphi\left(r-\gamma\right)$ which again implies that $n-\sigma\left(r-\rho\right)+\gamma^* < r-\gamma^*$, where we have used that $\varphi' < 0$. Simple rewriting yields: $\gamma^* < \dfrac{\sigma\left(r-\rho\right)+r-n}{2}$, which we know to be true from (P2.3) and (P2.4).

Similar reasoning does not, however, allow us to determine the validity of condition (iii). In order to study that condition we rely on numerical simulations, as discussed in the text.

C: Computation of Equilibrium Values of μ_C, μ_H, μ_Δ

From (3.10) we know that:

$$\mu_C\left(t-v_1\right) = \frac{\int_{t-D}^{t} \mu\left(t-v\right) p\left(t-v\right) C\left(v,t\right) dv}{\int_{t-D}^{t} p\left(t-v\right) C\left(v,t\right) dv} \tag{C.1}$$

Using the Euler equation, the fact that along the balanced growth path $C\left(v,v\right)/w\left(v\right)$ is independent of v, and the wage rate grows at the constant rate γ, we can write (C.1) as:

$$\frac{\int_{t-D}^{t} \mu\left(t-v\right) p\left(t-v\right) e^{\left(\sigma\left(r-\rho\right)-\gamma\right)\left(t-v\right)} dv}{\int_{t-D}^{t} p\left(t-v\right) e^{\left(\sigma\left(r-\rho\right)-\gamma\right)\left(t-v\right)} dv} \tag{C.2}$$

Recalling the demographic steady-state relationship (2.8) we can write (C.2) in the age domain as:

$$\frac{\int_0^D \mu(s) e^{-(n-\sigma(r-\rho)+\gamma)s-M(s)} ds}{\int_0^D e^{-(n-\sigma(r-\rho)+\gamma)s-M(s)} ds} \tag{C.3}$$

where $s = t - v$ is the age of the agent. This expression is seen to be independent of calendar time t.

Using the definitions of φ in (3.5) and $\mu(s)$ for the BCL demography gives:

$$\frac{\mu_1}{\mu_0 - 1} \frac{\int_0^D e^{-(n-\sigma(r-\rho)+\gamma-\mu_1)s} ds}{\varphi(n-\sigma(r-\rho)+\gamma)} \tag{C.4}$$

Evaluating the integral then gives:

$$\left(\frac{\mu_1}{\mu_0 - 1}\right)\left(\frac{1-e^{-(n-\sigma(r-\rho)+\gamma-\mu_1)D}}{n-\sigma(r-\rho)+\gamma-\mu_1}\right)\frac{1}{\varphi(n-\sigma(r-\rho)+\gamma)} \tag{C.5}$$

which for our specified parameters yields the value 0.0184.

From (B9.a) we know that:

$$\mu_H(\tau_1 - t) = \frac{\int_t^{t+D} \mu(\tau - t) w(\tau) e^{-r(\tau-t)-M(\tau-t)} d\tau}{\int_t^{t+D} w(\tau) e^{-r(\tau-t)-M(\tau-t)} d\tau} \tag{C.6}$$

Using the same arguments as above this can be shown to be independent of t and for the BCL function is:

$$\left(\frac{\mu_1}{\mu_0 - 1}\right)\left(\frac{1-e^{-(r-\gamma-\mu_1)D}}{r-\gamma-\mu_1}\right)\frac{1}{\varphi(r-\gamma)} \tag{C.7}$$

and assumes a simulated value of 0.0034. Similarly, from (B9.b) we obtain for the BCL function:

$$\left(\frac{\mu_1}{\mu_0 - 1}\right)\left(\frac{1-e^{-((1-\sigma)r+\sigma\rho-\mu_1)D}}{(1-\sigma)r+\sigma\rho-\mu_1}\right)\frac{1}{\varphi((1-\sigma)r+\sigma\rho)}. \tag{C.8}$$

which assumes a value of 0.0048.

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